

MULTISCALE ANALYSIS AND LOCALIZATION OF RANDOM OPERATORS

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ABSTRACT. A discussion of the method of multiscale analysis in the study of localization of random operators based on lectures given at *Random Schrödinger operators: methods, results, and perspectives*, États de la recherche, Université Paris 13, June 2002

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1. INTRODUCTION

In his seminal 1958 article [An1], Anderson argued that for a simple Schrödinger operator in a disordered medium, “at sufficiently low densities transport does not take place; the exact wave functions are localized in a small region of space.” This phenomenon, known as Anderson localization, originally studied in the context of quantum mechanical electrons in random media (e.g., [T]), was later found relevant also in the context of classical waves in random media (e.g., [An2, Ma, Jo1, Jo2]), where it was observed in light waves in an experiment conducted by Wiersma et al [WiBLR].

Anderson localization was initially given a spectral interpretation: pure point spectrum with exponentially decaying eigenstates (exponential localization). But the intuitive physical notion of localization has also a dynamical interpretation: the moments of a wave packet, initially localized both in space and in energy, should remain uniformly bounded under time evolution. (Dynamical localization implies pure point spectrum, but the converse is not true.) Although exponential localization has sometimes been called Anderson localization, we will use Anderson

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localization in a broader sense, since it can be argued the circle of ideas regarding localization, originating from [An1], include the physical notion of dynamical localization.

Localization for random operators was first established in the celebrated paper by Gol'dsheid, Molchanov and Pastur [GoMP] for a one dimensional continuous random Schrödinger operator. Their method was extended to other one and quasi-one (the strip) dimensional random Schrödinger operators [KuS, C, L]. But the multi-dimensional case required new methods.

The method with the wider applicability has been the multiscale analysis, a technique initially developed by Fröhlich and Spencer [FrS] and Fröhlich, Martinelli, Spencer and Scoppola [FrMSS], and simplified by von Dreifus [Dr] and von Dreifus and Klein [DrK]. (For the multiscale analysis per se, see also [HoM, Sp, DrK2, Kl1, Gr, Klo1, CoH1, FK3, KSS1, KSS2, Kr, St, GK1, GK4], for applications see also [CKM, KIMP, KILS, Klo2, Klo3, FK1, FK2, CoH2, FK4, W1, BCH1, BCH2, SVW, CoHT, Kl4, DeG, FiLM, Klo5, Z, DSS, U, KIK2, GK3, GK5, GK6].) Although it originally only gave exponential localization [FrMSS, DelyLS, SiW, DrK, CoH1], it was later shown to also yield dynamical localization by Germinet and De Bièvre [GD], strong dynamical localization for moments up to some finite order by Damanik and Stollman [DSt], and strong dynamical localization (up to all orders) in the Hilbert-Schmidt norm by Germinet and Klein [GK1]. The latest version of the multiscale analysis, the bootstrap multiscale analysis of Germinet and Klein [GK1], built out of four different multiscale analyses, yields exponential localization, semi-uniformly localized eigenfunctions (SULE), and sub-exponential decay of the expectation of the kernel of the evolution operator.

The other successful method for proving localization in the multi-dimensional case is the fractional moment method introduced by Aizenman and Molchanov [AM, A, ASFH], which has just been extended to the continuum by Aizenman et al [AENSS]. It yields exponential decay for the expectation of the kernel of the evolution operator, but it requires that the conditional expectation of certain random variables have bounded densities.

In these lectures we discuss the method of multiscale analysis in the study of localization of random operators. A random medium will be modeled by a ergodic random self-adjoint operator. In Section 2 we discuss the most important random operators: random Schrödinger operators, random Landau Hamiltonians, and random classical wave operators (Maxwell, acoustic, elastic). In Section 3 we discuss several definitions of localization from both the spectral and dynamical points of view. In Section 4 we describe the properties of random operators required by the multiscale analysis. In Section 5 we state and discuss the bootstrap multiscale analysis plus the four multiscale analyses used in its proof. In Section 6 we prove exponential and dynamical localization from the multiscale analysis. In Section 7 we show how to perform a multiscale analysis; we give a complete proof of the Dreifus-Klein multiscale analysis in the continuum.

These lectures were written in 2002. Since then Bourgain and Kenig [BouK] proved localization in the continuous Anderson-Bernoulli model, using a multiscale analysis. The Wegner estimate is established in the multiscale analysis using “free sites” and a new quantitative version of unique continuation which gives a lower bound on eigenfunctions. Since their Wegner estimate has weak probability estimates and the underlying random variables are discrete, they also introduced a

new method to prove Anderson localization from estimates on the finite-volume resolvents given by a single-energy multiscale analysis. The new method does not use spectral averaging as in [CoH1, DelyLS, SiW], which requires random variables with bounded densities. It is also not an energy-interval multiscale analysis as in [DrK, FrMSS], which requires better probability estimates. Subsequently, Germinet, Hislop and Klein [GHK1, GHK2, GHK3] proved localization for Schrödinger operators with Poisson random potential, using a multiscale analysis that exploits the probabilistic properties of Poisson point processes to control the randomness of the configurations, and at the same time allows the use of the new ideas introduced by Bourgain and Kenig.

2. RANDOM OPERATORS

Quantum and classical waves in random media are modeled by random self-adjoint operators on either $L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$ or $\ell^2(\mathbb{Z}^d; \mathbb{C}^n)$. Examples include:

- Random Schrödinger operators:

- ★ The Anderson model:

$$H_\omega = -\Delta + V_\omega \text{ on } \ell^2(\mathbb{Z}^d), \quad (2.1)$$

where Δ is the finite difference Laplacian and $\{V_\omega(x); x \in \mathbb{Z}^d\}$ are independent identically distributed bounded random variables. (E.g., [KuS, FrS, L, FrMSS, CKM, MS, KIMP, CyFKS, DrK, Sp, KILS, Kl1, Gr, AM, A, FK1, Kl2, Kl3, SVW, ASFH, W2, Klo4].)

- ★ Anderson Hamiltonians on the continuum:

$$H_\omega = -\Delta + V_{\text{per}} + V_\omega \text{ on } L^2(\mathbb{R}^d, dx), \quad (2.2)$$

where Δ is the Laplacian operator, V_{per} is a periodic potential (by rescaling we take the period to be one) of the form $V_{\text{per}} = V_{\text{per}}^{(1)} + V_{\text{per}}^{(2)}$, with $V_{\text{per}}^{(i)}$, $i = 1, 2$, periodic with period one, $0 \leq V_{\text{per}}^{(1)} \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$, $V_{\text{per}}^{(2)}$ relatively form-bounded with respect to $-\Delta$ with relative bound < 1 , and V_ω a random potential of the form

$$V_\omega(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} \omega_i u(x - i), \quad (2.3)$$

where $q \in \mathbb{N}$, $\omega = \{\omega_i; i \in \frac{1}{q}\mathbb{Z}^d\}$ are independent identically distributed bounded random variables, u is a real valued measurable function with compact support, $u \in L^p(\mathbb{R}^d, dx)$ with $p > \frac{d}{2}$ if $d \geq 2$ and $p = 2$ if $d = 1$. (E.g., [HoM, Klo1, Klo2, CoH1, Klo3, BCH1, KSS1, KSS2, GD, St, GK1, DSt, DSS, Klo5, Z, GK3, GK4, GK5, GK6, AENSS].)

- Random Landau Hamiltonians:

$$H_\omega = H_0 + V_\omega \text{ on } L^2(\mathbb{R}^2, dx), \quad (2.4)$$

where $H_0 = (-i\nabla - A)^2$, $A = \frac{B}{2}(x_2, -x_1)$ with $B > 0$, and the random potential V_ω is as in (2.3) with $q = 1$ and $u(x)$ bounded. (See [CoH2, W1, BCH2, GK4].)

- Random classical wave operators:

★ Maxwell operators in random media:

$$H_\omega = \frac{1}{\sqrt{\mu_\omega(x)}} \nabla \times \frac{1}{\varepsilon_\omega(x)} \nabla \times \frac{1}{\sqrt{\mu_\omega(x)}} \quad \text{on } L^2(\mathbb{R}^3, dx; \mathbb{C}^3) \quad (2.5)$$

where $\nabla \times$ is the operator given by the curl, $\varepsilon_\omega(x)$ is the random dielectric constant and $\mu_\omega(x)$ is the random magnetic permeability. We take

$$\varepsilon_\omega(x) = \varepsilon_0(x)\gamma_\omega(x), \text{ with } \gamma_\omega(x) = 1 + \sum_{i \in \frac{1}{q}\mathbb{Z}^3} \omega_i u(x-i), \quad (2.6)$$

$$\mu_\omega(x) = \mu_0(x)\beta_\omega(x), \text{ with } \beta_\omega(x) = 1 + \sum_{i \in \frac{1}{q}\mathbb{Z}^3} \omega_i v(x-i), \quad (2.7)$$

where $q \in \mathbb{N}$, $\omega = \{\omega_i; i \in \frac{1}{q}\mathbb{Z}^d\}$ are independent identically distributed bounded random variables taking values in the interval $[-1, 1]$, $\varepsilon_0(x)$ and $\mu_0(x)$ are periodic measurable functions (by rescaling we take the period to be one), such that $0 < \varepsilon_- \leq \varepsilon(x) \leq \varepsilon_+ < \infty$ and $0 < \mu_- \leq \mu(x) \leq \mu_+ < \infty$ for some constants ε_\pm and μ_\pm , $u(x)$ and $v(x)$ are nonnegative measurable real valued functions with compact support, such that

$$0 \leq U_- \leq U(x) \equiv \sum_{i \in \frac{1}{q}\mathbb{Z}^3} u_i(x) \leq U_+ < \infty, \quad (2.8)$$

$$0 \leq V_- \leq V(x) \equiv \sum_{i \in \frac{1}{q}\mathbb{Z}^3} v_i(x) \leq V_+ < \infty, \quad (2.9)$$

for some constants U_\pm and V_\pm , with $U_- + V_- > 0$ and $\max\{U_+, V_+\} < 1$. (See [FK2, FK4, Kl4, CoHT, KlK1, KlK2].)

★ Acoustic operators in random media:

$$H_\omega = \frac{1}{\sqrt{\kappa_\omega(x)}} \nabla^* \frac{1}{\rho_\omega(x)} \nabla \frac{1}{\sqrt{\kappa_\omega(x)}} \quad \text{on } L^2(\mathbb{R}^d, dx), \quad (2.10)$$

where ∇ is the gradient operator, and the random compressibility $\kappa_\omega(x)$ and the random mass density $\rho_\omega(x)$ are of the same form as $\varepsilon_\omega(x)$ and $\mu_\omega(x)$ in (2.6) and (2.7). (See [FK2, FK3, CoHT, KlK1, KlK2]).

★ Elastic operators in random media:

$$H_\omega = \frac{1}{\sqrt{\rho_\omega(x)}} \{ \nabla (\lambda_\omega(x) + 2\mu_\omega(x)) \nabla^* + \nabla \times \mu_\omega(x) \nabla \times \} \frac{1}{\sqrt{\rho_\omega(x)}} \quad (2.11)$$

on $L^2(\mathbb{R}^3, dx; \mathbb{C}^3)$, where the mass density $\rho_\omega(x)$, and the Lamé moduli $\lambda_\omega(x)$ and $\mu_\omega(x)$ are of the same form as $\varepsilon_\omega(x)$ and $\mu_\omega(x)$ in (2.6) and (2.7). (See [KIK1, KlK2]).

In all these examples the random operator H_ω is a \mathbb{Z}^d -ergodic random self-adjoint operator H_ω on a Hilbert space \mathcal{H} , where ω belongs to a set Ω with a probability measure \mathbb{P} and expectation \mathbb{E} , and either $\mathcal{H} = L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$ ("on the continuum") or $\mathcal{H} = \ell^2(\mathbb{Z}^d; \mathbb{C}^n)$ ("on the lattice"). They all satisfy the following definition.

Definition 2.1. An ergodic random operator is a \mathbb{Z}^d -ergodic measurable map H_ω from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (with expectation \mathbb{E}) to self-adjoint operators on either $L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$ or $\ell^2(\mathbb{Z}^d; \mathbb{C}^n)$.

By measurability of H_ω we mean that the mappings $\omega \rightarrow f(H_\omega)$ are weakly (and hence strongly) measurable for all bounded Borel measurable functions f on \mathbb{R} . (See [KM], [CL, Section V.1] for more details.) Random operators may be defined without any ergodicity requirement, ergodicity being an extra requirement, but since we will be dealing only with \mathbb{Z}^d -ergodic random operators, we included it in the definition for convenience. We recall that H_ω is \mathbb{Z}^d -ergodic if there exists a group representation of \mathbb{Z}^d by an ergodic family $\{\tau_y; y \in \mathbb{Z}^d\}$ of measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$U(y)H_\omega U(y)^* = H_{\tau_y(\omega)} \text{ for all } y \in \mathbb{Z}^d, \quad (2.12)$$

where $U(y)$ is the unitary operator given by translation: $(U(y)f)(x) = f(x - y)$. (Note that for Landau Hamiltonians translations are replaced by magnetic translations.)

An important consequence of ergodicity is that there exists a nonrandom set Σ such that $\sigma(H_\omega) = \Sigma$ with probability one, where $\sigma(A)$ denotes the spectrum of the operator A . In addition, the decomposition of $\sigma(H_\omega)$ into pure point spectrum $\sigma_{pp}(H_\omega)$, absolutely continuous spectrum $\sigma_{ac}(H_\omega)$, and singular continuous spectrum $\sigma_{sc}(H_\omega)$ is also independent of the choice of ω with probability one, i.e., there are nonrandom sets Σ_{pp} , Σ_{ac} and Σ_{sc} , such that $\sigma_{pp}(H_\omega) = \Sigma_{pp}$, $\sigma_{ac}(H_\omega) = \Sigma_{ac}$, and $\sigma_{sc}(H_\omega) = \Sigma_{sc}$ with probability one. (See [P, KuS, KM, PF, CL, CyFKS].)

3. SPECTRAL AND DYNAMICAL LOCALIZATION

Localization can be interpreted from either the spectral or the dynamical point of views. We give selected definitions from each point of view.

By χ_B we denote the characteristic function of the set $B \subset \mathbb{R}^d$ (or \mathbb{Z}^d). By χ_x we denote the characteristic function of the cube of side 1 centered at $x \in \mathbb{Z}^d$. We write $\langle x \rangle = \sqrt{1 + |x|^2}$. The spectral projection of H_ω is denoted by $E_\omega(\cdot)$. The Hilbert-Schmidt norm of an operator A is written as $\|A\|_2$.

Definition 3.1. Let H_ω be an ergodic random operator and \mathcal{I} an open interval. Then

- (i) H_ω exhibits spectral localization (SL) in \mathcal{I} if it has pure point spectrum in \mathcal{I} , i.e., $\Sigma \cap \mathcal{I} = \Sigma_{pp} \cap \mathcal{I} \neq \emptyset$ and $\Sigma_{ac} \cap \mathcal{I} = \Sigma_{sc} \cap \mathcal{I} = \emptyset$.
- (ii) H_ω exhibits exponential localization (EL) in \mathcal{I} if it exhibits spectral localization in \mathcal{I} and for \mathbb{P} -almost every ω the eigenfunctions of H_ω with eigenvalue in \mathcal{I} decay exponentially in the L^2 -sense. (A function ψ decays exponentially in the L^2 -sense if $\|\chi_x \psi\|$ decays exponentially, i.e., $\|\chi_x \psi\| \leq C e^{-m|x|}$ with C and $m > 0$ constants.)
- (iii) H_ω exhibits dynamical localization (DL) in \mathcal{I} if $\Sigma \cap \mathcal{I} \neq \emptyset$ and, for \mathbb{P} -almost every ω , each compact interval $I \subset \mathcal{I}$, and $\psi \in \mathcal{H}$ with compact support, we have

$$\sup_{t \in \mathbb{R}} \left\| \langle x \rangle^{\frac{n}{2}} E_\omega(I) e^{-itH_\omega} \psi \right\| < \infty \text{ for all } n \geq 0. \quad (3.1)$$

(iv) H_ω exhibits strong dynamical localization (SDL) in \mathcal{I} if $\Sigma \cap \mathcal{I} \neq \emptyset$ and for each compact interval $I \subset \mathcal{I}$ and $\psi \in \mathcal{H}$ with compact support, we have

$$\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left\| \langle x \rangle^{\frac{n}{2}} E_\omega(I) e^{-itH_\omega} \psi \right\|^2 \right\} < \infty \quad \text{for all } n \geq 0. \quad (3.2)$$

(v) H_ω exhibits strong HS-dynamical localization (SHSDL) in \mathcal{I} if $\Sigma \cap \mathcal{I} \neq \emptyset$ and for each compact interval $I \subset \mathcal{I}$ and bounded Borel set B we have

$$\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left\| \langle x \rangle^{\frac{n}{2}} E_\omega(I) e^{-itH_\omega} \chi_B \right\|_2^2 \right\} < \infty \quad \text{for all } n \geq 0. \quad (3.3)$$

(vi) H_ω exhibits strong full HS-dynamical localization (SFHSDL) in \mathcal{I} if $\Sigma \cap \mathcal{I} \neq \emptyset$ and for each compact interval $I \subset \mathcal{I}$ and bounded Borel set B we have

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \left\| \langle x \rangle^{\frac{n}{2}} E_\omega(I) f(H_\omega) \chi_B \right\|_2^2 \right\} < \infty \quad \text{for all } n \geq 0, \quad (3.4)$$

the supremum being taken over all Borel functions f of a real variable, with $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$

(vii) H_ω exhibits strong sub-exponential HS-kernel decay (SSEHSDK) in \mathcal{I} if $\Sigma \cap \mathcal{I} \neq \emptyset$ and for each compact interval $I \subset \mathcal{I}$ and $0 < \zeta < 1$ there is a finite constant $C_{I,\zeta}$ such that

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \left\| \chi_x E_\omega(I) f(H_\omega) \chi_y \right\|_2^2 \right\} \leq C_{I,\zeta} e^{-|x-y|^\zeta}, \quad (3.5)$$

for all $x, y \in \mathbb{Z}^d$, the supremum being taken over all Borel functions f of a real variable, with $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$.

Definition 3.2. Let H_ω be an ergodic random operator. The spectral localization region Σ_{SL} , exponential localization region Σ_{EL} , dynamical localization region Σ_{DL} , strong dynamical localization region Σ_{SDL} , strong HS-dynamical localization region Σ_{SHSDL} , strong full HS-dynamical localization region Σ_{SFHSDL} , strong sub-exponential HS-kernel decay region Σ_{SSEHSDK} , for the random operator H_ω , are defined as the set of $E \in \Sigma$ for which there exists some open interval $\mathcal{I} \ni E$ such that H_ω exhibits spectral localization, exponential localization, dynamical localization, strong dynamical localization, strong HS-dynamical localization, strong full HS-dynamical localization region, strong sub-exponential HS-kernel decay, respectively, in \mathcal{I} .

Remark 3.3. Note that

$$\Sigma_{\text{SSEHSDK}} \subset \Sigma_{\text{SFHSDL}} \subset \Sigma_{\text{SHSDL}} \subset \Sigma_{\text{SDL}} \subset \Sigma_{\text{DL}} \subset \Sigma_{\text{SL}}. \quad (3.6)$$

That $\Sigma_{\text{SSEHSDK}} \subset \Sigma_{\text{SFHSDL}}$ is a simple calculation (see [GK1, Proof of Corollary 3.10]); that $\Sigma_{\text{SFHSDL}} \subset \Sigma_{\text{SHSDL}} \subset \Sigma_{\text{SDL}} \subset \Sigma_{\text{DL}}$ is obvious; that $\Sigma_{\text{DL}} \subset \Sigma_{\text{SL}}$ follows from the RAGE Theorem (e.g., the argument in [CyFKS, Theorem 9.21]). But dynamical localization is actually a strictly stronger notion than pure point spectrum, since the latter can take place whereas a quasi-ballistic motion is observed [DelJLS].

For an ergodic random operator with suitable properties, spelled out in the next section, the original multiscale analyses showed that decay of the resolvent in a finite, but large enough, volume with high probability (the “starting hypothesis”

for the multiscale analysis)) gave a sufficient condition for $E \in \Sigma_{\text{SL}}$ [FrS, FrMSS, Dr, DrK]. Later that condition was shown to be sufficient for $E \in \Sigma_{\text{DL}}$ [GD], $E \in \Sigma_{\text{SDL}}$ [DSt] (more precisely, they show that (3.3) holds with the operator norm substituted for the Hilbert-Schmidt norm and $n \leq n_0$ for some $n_0 < \infty$), and finally $E \in \Sigma_{\text{SSEHSKD}}$ [GK1]. Moreover, the converse was found to be true: $E \in \Sigma_{\text{SHSDL}}$ implies the starting hypothesis of the multiscale analysis [GK3].

Remark 3.4. *The multiscale analysis region Σ_{MSA} is given in Definition 5.3 as the region where the conclusions of the multiscale analysis hold. If the ergodic random operator satisfies the requirements of the multiscale analysis in an open interval \mathcal{I} , it will be shown in Theorem 6.1 that $\Sigma_{\text{MSA}} \cap \mathcal{I} \subset \Sigma_{\text{EL}} \cap \Sigma_{\text{SSEHSKD}} \cap \mathcal{I}$. If in addition we have property (4.17) and the kernel decay estimates of [GK2] hold uniformly for \mathbb{P} -a.e. ω (both requirements are usually satisfied), then it is proven in [GK3] that*

$$\Sigma_{\text{MSA}} \cap \mathcal{I} = \Sigma_{\text{SSEHSKD}} \cap \mathcal{I} = \Sigma_{\text{SHSDL}} \cap \mathcal{I}. \quad (3.7)$$

Moreover, in [GK7] it is shown that the spectral region in (3.7) has characterizations by the decay of eigenfunction correlations and by the decay of Fermi projections, and that the former implies finite multiplicity of the eigenvalues of the ergodic random operator.

4. REQUIREMENTS OF THE MULTISCALE ANALYSIS

We now state the properties of the ergodic random operator H_ω that are required for the multiscale analysis and its consequence. We will work on the continuum, but everything will work on the lattice (easier case) with appropriate modifications. We fix an open interval \mathcal{I} .

4.1. Generalized eigenfunction expansion. Generalized eigenfunction expansions were originally developed for elliptic partial differential operators with smooth coefficients (see Berezanskii [Be] and references therein). These expansions were extended to Schrödinger operators with singular potentials by Simon [Si] (see also references therein), and to classical wave operators with nonsmooth coefficients by Klein, Koines and Seifert [KIKS].

These expansions construct polynomially bounded generalized eigenfunctions for a set of generalized eigenvalues with full spectral measure. These generalized eigenfunctions were used by Pastur [P] and by Martinelli and Scoppola [MS] to prove that certain Schrödinger operators with random potentials have no absolutely continuous spectrum. They played a crucial role in the work by Fröhlich, Martinelli, Spencer and Scoppola [FrMSS] and by von Dreifus and Klein [DrK] on exponential localization of random Schrödinger operators, providing the crucial link between the multiscale analysis and pure point spectrum: the exponential decay of finite volume Green's functions (obtained by a multiscale analysis) forces polynomially bounded generalized eigenfunctions to be bona fide eigenfunctions, so the spectrum is at most countable and hence pure point.

In [GK1], as in [G, GJ], the generalized eigenfunction expansion itself (not just the existence of polynomially bounded generalized eigenfunctions) is used to provide the link between the multiscale analysis and strong HS-dynamical localization (and hence pure point spectrum).

We will now state the properties of an ergodic random operator that guarantees the existence of a generalized eigenfunction expansion. We follow the approach in [KIKS, Section 3].

Let $\mathcal{H} = L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$. (We discuss the generalized eigenfunction expansion on the continuum, but an analogous discussion is valid on the lattice.) Given $\nu > d/4$ (omitted from the notation), we define the weighted spaces \mathcal{H}_\pm :

$$\mathcal{H}_\pm = L^2(\mathbb{R}^d, \langle x \rangle^{\pm 4\nu} dx; \mathbb{C}^n). \quad (4.1)$$

\mathcal{H}_- is a space of polynomially L^2 -bounded functions. (Recall $\langle x \rangle = \sqrt{1 + |x|^2}$.) The sesquilinear form

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}_+, \mathcal{H}_-} = \int \overline{\phi_1(x)} \cdot \phi_2(x) dx, \quad (4.2)$$

where $\phi_1 \in \mathcal{H}_+$ and $\phi_2 \in \mathcal{H}_-$, makes \mathcal{H}_+ and \mathcal{H}_- conjugate duals to each other. By O^\dagger we will denote the adjoint of an operator O with respect to this duality. By construction, $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$, the natural injections $\iota_+ : \mathcal{H}_+ \rightarrow \mathcal{H}$ and $\iota_- : \mathcal{H} \rightarrow \mathcal{H}_-$ being continuous with dense range, with $\iota_+^\dagger = \iota_-$.

We set T to be the self-adjoint operator on \mathcal{H} given by multiplication by the function $\langle x \rangle^{2\nu}$; note that T^{-1} is bounded. The operators $T_+ : \mathcal{H}_+ \rightarrow \mathcal{H}$ and $T_- : \mathcal{H} \rightarrow \mathcal{H}_-$, defined by $T_+ = T\iota_+$, T_- the closure of the operator ι_-T on $\mathcal{D}(T)$, are unitary with $T_- = T_+^\dagger$. The map $\tau : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$, with $\tau(C) = T_- C T_+$, is a Banach space isomorphism, as T_\pm are unitary operators. ($\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the Banach space of bounded operators from \mathcal{H}_1 to \mathcal{H}_2 , $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$.) If $1 \leq q < \infty$, we define $\mathcal{T}_q(\mathcal{H}_+, \mathcal{H}_-) = \tau(\mathcal{T}_q(\mathcal{H}))$, where $\mathcal{T}_q(\mathcal{H})$ denotes the Banach space of bounded operators S on \mathcal{H} with $\|S\|_q = (\text{tr } |S|^q)^{\frac{1}{q}} < \infty$. By construction, $\mathcal{T}_q(\mathcal{H}_+, \mathcal{H}_-)$, equipped with the norm $\|B\|_q = \|\tau^{-1}(B)\|_q$, is a Banach space isomorphic to $\mathcal{T}_q(\mathcal{H})$, with $\mathcal{T}_2(\mathcal{H}_+, \mathcal{H}_-)$ being the usual Hilbert space of Hilbert-Schmidt operators from \mathcal{H}_+ to \mathcal{H}_- .

Note that

$$\|\chi_x\|_{\mathcal{H}, \mathcal{H}_+} = \|\chi_x\|_{\mathcal{H}_-, \mathcal{H}} \leq \left(\frac{3}{2}\right)^\nu \langle x \rangle^{2\nu} \quad (4.3)$$

for all $x \in \mathbb{R}^d$. (Given an operator $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, $\|B\|_{\mathcal{H}_1, \mathcal{H}_2}$ will denote its operator norm.)

The following property guarantees the existence of a generalized eigenfunction expansion (GEE) in the open interval \mathcal{I} with the right properties (see [KIKS, Section 3] for details). We write $E_\omega(B)$ for the spectral projections of the operator H_ω , i.e., $E_\omega(J) = \chi_J(H_\omega)$ for any bounded Borel set $J \subset \mathbb{R}$. We will fix an appropriate $\nu > d/4$ and use the corresponding operator T and weighted spaces \mathcal{H}_\pm as in (4.1).

(GEE) *For some $\nu > d/4$ the set*

$$\mathcal{D}_+^\omega = \{\phi \in \mathcal{D}(H_\omega) \cap \mathcal{H}_+; H_\omega \phi \in \mathcal{H}_+\} \quad (4.4)$$

is dense in \mathcal{H}_+ and an operator core for H_ω with probability one. Moreover, there exists a bounded, continuous function f on \mathbb{R} , strictly positive on the spectrum of H_ω , such that

$$\text{tr}_\mathcal{H} (T^{-1} f(H_\omega) E_\omega(\mathcal{I}) T^{-1}) < \infty \quad (4.5)$$

with probability one.

A measurable function $\psi : \mathbb{R}^d \rightarrow \mathbb{C}^n$ is said to be a generalized eigenfunction of H_ω with generalized eigenvalue λ , if $\psi \in \mathcal{H}_- \setminus \{0\}$ and

$$\langle H_\omega \phi, \psi \rangle_{\mathcal{H}_+, \mathcal{H}_-} = \lambda \langle \phi, \psi \rangle_{\mathcal{H}_+, \mathcal{H}_-} \quad \text{for all } \phi \in \mathcal{D}_+^\omega.$$

It follows from the first part of property (GEE) that if a generalized eigenfunction is in \mathcal{H} , then it is a bona fide eigenfunction.

If (GEE) holds, the following is true for \mathbb{P} -almost every ω : For all bounded Borel sets J we have

$$\mathrm{tr}_{\mathcal{H}} (T^{-1} E_{\omega}(J \cap \mathcal{I}) T^{-1}) < +\infty, \quad (4.6)$$

and hence

$$\mu_{\omega}(J) = \mathrm{tr}_{\mathcal{H}} (T^{-1} E_{\omega}(J \cap \mathcal{I}) T^{-1}) \quad (4.7)$$

is a spectral measure for the restriction of H_{ω} to the Hilbert space $E_{\omega}(\mathcal{I})\mathcal{H}$, with

$$\mu_{\omega}(J) < \infty \text{ for } J \text{ bounded.} \quad (4.8)$$

In particular, we have a generalized eigenfunction expansion for H_{ω} : with probability one, there exists a μ_{ω} -locally integrable function $P_{\omega}(\lambda)$ from the real line into $\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$, with

$$P_{\omega}(\lambda) = P_{\omega}(\lambda)^{\dagger} \quad (4.9)$$

and

$$\mathrm{tr}_{\mathcal{H}} (T_{-}^{-1} P_{\omega}(\lambda) T_{+}^{-1}) = 1 \text{ for } \mu_{\omega} - \text{a.e. } \lambda, \quad (4.10)$$

such that

$$\iota_{-} E_{\omega}(J \cap \mathcal{I}) \iota_{+} = \int_J P_{\omega}(\lambda) d\mu_{\omega}(\lambda) \text{ for bounded Borel sets } J, \quad (4.11)$$

where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)$ -valued functions. Moreover, for μ_{ω} -almost every λ , if $\phi \in \mathcal{H}_+$ and $P_{\omega}(\lambda)\phi \neq 0$, then $P_{\omega}(\lambda)\phi$ is a generalized eigenfunction of H_{ω} with generalized eigenvalue λ . It follows, using (4.11), that μ_{ω} -almost every λ is a generalized eigenvalue of H_{ω} .

Lemma 4.1. *If the ergodic random operator H_{ω} has property (GEE), then for \mathbb{P} -almost every ω , we have*

$$\|\chi_x P_{\omega}(\lambda) \chi_y\|_1 \leq \left(\frac{3}{2}\right)^{2\nu} \langle x \rangle^{2\nu} \langle y \rangle^{2\nu} \quad (4.12)$$

for all $x, y \in \mathbb{R}^d$ and μ_{ω} -almost every λ . ($\|\cdot\|_1$ denotes the trace norm in \mathcal{H} .)

Proof. Since

$$\|\chi_x P_{\omega}(\lambda) \chi_y\|_1 \leq \|\chi_x\|_{\mathcal{H}_-, \mathcal{H}} \|P_{\omega}(\lambda)\|_{\mathcal{T}_1(\mathcal{H}_+, \mathcal{H}_-)} \|\chi_y\|_{\mathcal{H}, \mathcal{H}_+}, \quad (4.13)$$

(4.12) follows from (4.3) and (4.10). \square

(GEE) suffices for proofs of exponential localization [FrMSS, DrK] and dynamical localization [GD, G]. But for strong dynamical localization we need to strengthen (4.5).

(SGEE) *Property (GEE) holds with*

$$\mathbb{E} \left\{ \left[\mathrm{tr}_{\mathcal{H}} (T^{-1} f(H_{\omega}) E_{\omega}(\mathcal{I}) T^{-1}) \right]^2 \right\} < \infty. \quad (4.14)$$

It follows that

$$\mathbb{E} \left\{ \left[\mathrm{tr}_{\mathcal{H}} (T^{-1} E_{\omega}(J \cap \mathcal{I}) T^{-1}) \right]^2 \right\} < \infty \quad (4.15)$$

for all bounded Borel sets J , so we have a stronger version of (4.8):

$$\mathbb{E} \left\{ [\mu_{\omega}(J)]^2 \right\} < \infty \text{ for } J \text{ bounded.} \quad (4.16)$$

Remark 4.2. Estimate (4.14) is true for the usual ergodic random operators. In fact one usually proves the stronger

$$\| \operatorname{tr}_{\mathcal{H}} (T^{-1} f(H_\omega) E_\omega(\mathcal{I}) T^{-1}) \|_{L^\infty(\Omega, \mathcal{F}, \mathbb{P})} < \infty, \quad (4.17)$$

which is a hypothesis in [DSt]. For a proof, see [KIKS, Theorem 1.1] for classical wave operators and [Si],[GK3, Theorem A.1] for Schrödinger operators.

4.2. Finite volume operators and their properties. Throughout these lectures we use the sup norm in \mathbb{R}^d :

$$|x| = \max\{|x_i|, i = 1, \dots, d\}. \quad (4.18)$$

By $\Lambda_L(x)$ we denote the open box (or cube) of side $L > 0$ centered at $x \in \mathbb{R}^d$:

$$\Lambda_L(x) = \{y \in \mathbb{R}^d; |y - x| < \frac{L}{2}\}, \quad (4.19)$$

and by $\overline{\Lambda}_L(x)$ the closed box. We set

$$\chi_{x,L} = \chi_{\Lambda_L(x)}, \quad \chi_x = \chi_{x,1} = \chi_{\Lambda_1(x)}. \quad (4.20)$$

We will usually take boxes centered at sites $x \in \mathbb{Z}^d$ with side $L \in 2\mathbb{N}$. Given such a box $\Lambda_L(x)$, we set

$$\Upsilon_L(x) = \{y \in \mathbb{Z}^d; |y - x| = \frac{L}{2} - 1\}, \quad (4.21)$$

and define its boundary belt by

$$\tilde{\Upsilon}_L(x) = \overline{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x) = \bigcup_{y \in \Upsilon_L(x)} \overline{\Lambda}_1(y); \quad (4.22)$$

it has the characteristic function

$$\Gamma_{x,L} = \chi_{\tilde{\Upsilon}_L(x)} = \sum_{y \in \Upsilon_L(x)} \chi_y \text{ a.e.} \quad (4.23)$$

Note that

$$|\Upsilon_L(x)| = (L-1)^d - (L-2)^d = d \int_{L-2}^{L-1} x^{d-1} dx \leq d(L-1)^{d-1}. \quad (4.24)$$

We shall suppress the dependency of a box on its center when not necessary. When using boxes Λ_ℓ contained in bigger boxes Λ_L , we shall need to know that the small box is inside the belt $\tilde{\Upsilon}_L$ of the bigger one. If $L > \ell + 3$ and $x \in \mathbb{Z}^d$, we say that

$$\Lambda_\ell \subset \Lambda_L(x) \quad \text{if} \quad \Lambda_\ell \subset \Lambda_{L-3}(x). \quad (4.25)$$

Very often we will require $L \in 6\mathbb{N}$; given $K \geq 6$, we set

$$[K]_{6\mathbb{N}} = \max\{L \in 6\mathbb{N}; L \leq K\}. \quad (4.26)$$

The multiscale analysis requires the notion of a *finite volume operator*, a “restriction” $H_{\omega,x,L}$ of H_ω to the box $\Lambda_L(x)$ where the “randomness based outside the box $\Lambda_L(x)$ ” is not taken into account. Usually $H_{\omega,x,L}$ is defined as the restriction of H_ω , either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition. The operator $H_{\omega,x,L}$ then acts on $L^2(\Lambda_L(x), dx; \mathbb{C}^n)$. But $H_{\omega,x,L}$ may also be defined as acting on the whole space, by throwing away the random coefficients “based outside the box $\Lambda_L(x)$ ”; this is usually done for random Landau operators [CoH2, W1, GK4]. In all cases the finite volume operators have either compact resolvent or are relatively compact perturbations of the free Hamiltonian.

Definition 4.3. *The ergodic random operator H_ω is called standard if it has a finite volume restriction, i.e., if for each $x \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$ there is a measurable map $H_{\omega,x,L}$ from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to self-adjoint operators on $L^2(\Lambda_L(x), dx; \mathbb{C}^n)$ (or all such mappings taking values as self-adjoint operators on $L^2(\mathbb{R}^d, dx; \mathbb{C}^n)$), such that*

$$U(y)H_{\omega,x,L}U(y)^* = H_{\tau_y(\omega),x+y,L} \text{ for all } y \in \mathbb{Z}^d, \quad (4.27)$$

where $U(y)$ is as in (2.12). We write $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ for the resolvent of the finite volume operator $H_{\omega,x,L}$ and $E_{\omega,x,L}(\cdot)$ for its spectral projection.

The multiscale analysis and its consequences require certain properties of the finite volume restriction of the ergodic random operator. These properties are routinely verified for the usual ergodic random operators (e.g., [FrS, FrMSS, DrK, HoM, CoH1, CoH2, FK3, FK4, W1, St, KlK1, KlK2, GK3, GK4].

The first property is *independence at a distance* (IAD) for the finite volume operators. It says that if boxes are far apart, events defined by the restrictions of the random operator H_ω to these boxes are independent. This assumption can be relaxed in some ways by suitable modifications of the multiscale analysis (e.g., [DrK2, KSS2, FiLM, Z]).

An event is said to be based on the box $\Lambda_L(x)$ if it is determined by conditions on the finite volume operator $H_{\omega,x,L}$. Given $\varrho > 0$, we say that two boxes $\Lambda_L(x)$ and $\Lambda_{L'}(x')$ are ϱ -nonoverlapping if $|x - x'| > \frac{L+L'}{2} + \varrho$, i.e., if $\text{dist}(\Lambda_L(x), \Lambda_{L'}(x')) > \varrho$.

(IAD) *There exists $\varrho > 0$ such that events based on ϱ -nonoverlapping boxes are independent.*

The remaining properties are to hold in the fixed open interval \mathcal{I} .

The first such property is reminiscent of the Simon-Lieb inequality (SLI) in Classical Statistical Mechanics. It relates resolvents in different scales. In the lattice it is an immediate consequence of the resolvent identity, in this context it was originally used in [FrS]. In the continuum, its proof requires interior estimates, and was proved in [CoH1] for Schrödinger operators. It was adapted to classical wave operators in [FK3]. We state it in the form given in [KlK1, Lemma 3.8] for classical wave operators and [GK3, Theorem A.1] for Schrödinger operators. (The lattice requires slight modifications.)

(SLI) *For any compact interval $I \subset \mathcal{I}$ there exists a finite constant γ_I , such that, given $L, \ell', \ell'' \in 2\mathbb{N}$, $x, y, y' \in \mathbb{Z}^d$ with $\Lambda_{\ell''}(y) \subset \Lambda_{\ell'}(y') \subset \Lambda_L(x)$, then for \mathbb{P} -almost every ω , if $E \in I$ with $E \notin \sigma(H_{\omega,x,L}) \cup \sigma(H_{\omega,y',\ell'})$, we have*

$$\|\Gamma_{x,L}R_{\omega,x,L}(E)\chi_{y,\ell''}\| \leq \gamma_I \|\Gamma_{y',\ell'}R_{\omega,y',\ell'}(E)\chi_{y,\ell''}\| \|\Gamma_{x,L}R_{\omega,x,L}(E)\Gamma_{y',\ell'}\|. \quad (4.28)$$

Remark 4.4. *Property (SLI) will be used in the following way: We will take $\ell'' = \frac{\ell}{3}$ with $\ell \in 6\mathbb{N}$, and $\ell' = k\frac{\ell}{3}$ with $3 \leq k \in \mathbb{N}$. By a cell we will mean a closed box $\overline{\Lambda}_{\frac{\ell}{3}}(y'')$, with $y'' \in \frac{\ell}{6}\mathbb{Z}^d$. We define \mathbb{Z}_{even} and \mathbb{Z}_{odd} to be the sets of even and odd integers. We take $y \in \frac{\ell}{6}\mathbb{Z}^d$, so $\chi_{y,\frac{\ell}{3}}$ is the characteristic function of a cell. We want the closed box $\overline{\Lambda}_{\ell'}(y')$ to be exactly covered by cells (in effect, by k^d cells); thus we specify $y' \in \frac{\ell}{3}\mathbb{Z}^d = \frac{\ell}{6}\mathbb{Z}_{\text{even}}^d$ if k is odd, and $y' \in \frac{\ell}{3}\mathbb{Z}^d + \frac{\ell}{6}(1, 1, \dots, 1) = \frac{\ell}{6}\mathbb{Z}_{\text{odd}}^d$ if*

k is even. We then replace the boundary belt $\tilde{\Upsilon}_{\ell'}(y')$ (of width 1) by a thicker belt $\tilde{\Upsilon}_{\ell',\ell}(y')$ of width $\frac{\ell}{3}$. To do so, we set

$$\Upsilon_{\ell',\ell}(y') = \left\{ y'' \in \frac{\ell}{3}\mathbb{Z}^d; |y'' - y'| = \frac{\ell'}{2} - \frac{\ell}{6} \right\}, \quad (4.29)$$

and define the boundary ℓ -belt of $\Lambda_{\ell'}(y')$ by

$$\tilde{\Upsilon}_{\ell',\ell}(y') = \overline{\Lambda}_{\ell'}(y') \setminus \Lambda_{\ell' - \frac{2\ell}{3}}(y') = \bigcup_{y'' \in \Upsilon_{\ell',\ell}(y')} \overline{\Lambda}_{\frac{\ell}{3}}(y''), \quad (4.30)$$

with characteristic function

$$\Gamma_{y',\ell',\ell} = \chi_{\tilde{\Upsilon}_{\ell',\ell}(y')} = \sum_{y'' \in \Upsilon_{\ell',\ell}(y')} \chi_{y'', \frac{\ell}{3}} \text{ a.e.} \quad (4.31)$$

Note that

$$|\Upsilon_{\ell',\ell}(y')| = (k^d - (k-2)^d) \leq k^d. \quad (4.32)$$

Since $\Gamma_{y',\ell',\ell} \Gamma_{y',\ell'} = \Gamma_{y',\ell'}$, the projection $\Gamma_{\ell'}$ on the belt of $\Lambda_{\ell'}$ can be replaced by the projection over the thicker belt of width $\frac{\ell}{3}$, which can be decomposed in boxes of side $\frac{\ell}{3}$. Thus (4.28) yields

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{y, \frac{\ell}{3}}\| \leq k^d \gamma_I \|\Gamma_{y',\ell'} R_{\omega,y',\ell'}(E) \chi_{y, \frac{\ell}{3}}\| \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{y'', \frac{\ell}{3}}\| \quad (4.33)$$

for some $y'' \in \Upsilon_{\ell',\ell}(y')$. Performing the SLI, i.e., using the estimate (4.33), we moved from the cell center y to the cell center y'' .

Remark 4.5. While performing a multiscale analysis we will use (4.33) with either $\ell' = \ell$ (for good boxes), or some $\ell' = k\frac{\ell}{3}$, $k > 3$, which will be the side of a bad box. Note that in the first case, $k = 3$, and the geometric factor is $3^d - 1 \leq 3^d$. In that case note also that we must have $y = y'$ and $|y'' - y| = \frac{\ell}{3}$, so after performing the SLI we moved to an adjacent cell center, i.e., by $\frac{\ell}{3}$ in the sup norm. (Recall that we are using the sup norm in \mathbb{R}^d , so we may move both sidewise and along the diagonals.)

The second property is an estimate of generalized eigenfunctions in terms of finite volume resolvents. It is not needed for the multiscale analysis, but it plays an important role in obtaining localization from the multiscale analysis [FrMSS, DrK, FK3, GK1]. We call it an *eigenfunction decay inequality* (EDI), since it translates decay of finite volume resolvents into decay of generalized eigenfunctions ; we present it as proved in [KLK1, Lemma 3.9] and [GK3, Theorem A.1]. It is closely related to property (SLI), the proofs being very similar.

(EDI) For any compact interval $I \subset \mathcal{I}$ there exists a finite constant $\tilde{\gamma}_I$, such that for \mathbb{P} -almost every ω , given a generalized eigenfunction ψ of H_ω with generalized eigenvalue $E \in I$, we have for any $x \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$ with $E \notin \sigma(H_{\omega,x,L})$ that

$$\|\chi_x \psi\| \leq \tilde{\gamma}_I \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_x\| \|\Gamma_{x,L} \psi\|. \quad (4.34)$$

Typically we have $\tilde{\gamma}_I = \gamma_I$, with γ_I as in (4.28). We will use the following consequence of (4.34):

$$\|\chi_x \psi\| \leq d \tilde{\gamma}_I L^{d-1} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_x\| \|\chi_y \psi\| \quad (4.35)$$

for some $y \in \Upsilon_L(x)$.

The third property is an “a priori” estimate on the average *number of eigenvalues* (NE) of finite volume random operators in a fixed, bounded interval. It is usually proved by a deterministic argument, using the well known bound for the Laplacian [CoH1, FK3, FK4, KIK1]. It is, of course, entirely obvious in the lattice.

(NE) *For any compact interval $I \subset \mathcal{I}$ there exists a finite constant C_I such that*

$$\mathbb{E}(\text{tr}_{\mathcal{H}} E_{\omega,x,L}(I)) \leq C_I L^d \quad (4.36)$$

for all $x \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$.

The final property is a form of *Wegner’s estimate* (W), a probabilistic estimate on the size of the resolvent. It is a crucial ingredient for the multiscale analysis, where it is used to control the bad regions.

(W) *For some $b \geq 1$ there exists a finite constant Q_I for each compact interval $I \subset \mathcal{I}$, such that*

$$\mathbb{P}\{\text{dist}(\sigma(H_{\omega,x,L}), E) \leq \eta\} \leq Q_I \eta L^{bd}, \quad (4.37)$$

for all $E \in I$, $0 < \eta \leq 1$, $x \in \mathbb{Z}^d$, and $L \in 2\mathbb{N}$.

Remark 4.6. *In practice we have either $b = 1$ or $b = 2$ in the Wegner estimate (4.37). For some random Schrödinger operators with Anderson potential we may have $b = 1$ [CoH1, Klo3] (including the Landau Hamiltonian). For classical waves in random media, (4.37) has been proven with $b = 2$ [FK3, FK4, KIK2]. More recently the correct volume dependency (i.e., $b = 1$) was obtained in [CoHN, CoHKN, HK] for random Schrödinger operators, at the price of losing a bit in the η dependency; more precisely, the right hand side of (4.37) is replaced by $Q_{a,I} \eta^a L^d$ for any $0 < a < 1$. In these lectures, we shall use (4.37) as stated, the modifications in our methods required for the other forms of (4.37) being obvious. Our methods may also accomodate properties (NE) and (W) being valid only for large L , and/or property (W) being valid only for $\eta < \eta_L$ for some appropriate η_L , say $\eta_L = L^{-r}$, some $r > 0$, or $\eta_L = e^{-L^\beta}$ for some $0 < \beta < 1$. The latter is of importance if one wants to deal with singular probability measures like Bernoulli [CKM, KILS, DeG, DSS].*

Remark 4.7. *In the continuum one usually proves the stronger estimate [HoM, CoH1, CoH2, FK3, FK4, KIK2, CoHN]:*

$$\mathbb{E}(\text{tr}_{\mathcal{H}} E_{H_{\omega,x,L}}([E - \eta, E + \eta])) \leq Q_I \eta L^{bd}, \quad (4.38)$$

from which (4.37) follows by Chebychev’s inequality. The estimate (4.36) is used as an “a priori” estimate in the proof of (4.38).

5. THE BOOTSTRAP MULTISCALE ANALYSIS

Given a standard ergodic random operator H_ω , the multiscale analysis looks for localization by studying the probability of decay of the finite volume resolvent from the center of a box $\Lambda_L(x)$ to its boundary belt as measured by

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,\frac{L}{3}}\|. \quad (5.1)$$

We start with three definitions, which characterize “good boxes” in a given scale by different types of decay relative to the scale.

Definition 5.1. *Given $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, with $E \notin \sigma(H_{\omega,x,L})$, we say that the box $\Lambda_L(x)$ is*

(i) (ω, θ, E) -suitable for a given $\theta > 0$ if

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,\frac{L}{3}}\| \leq \frac{1}{L^\theta}. \quad (5.2)$$

(ii) (ω, ζ, E) -sub-exponentially-suitable for a given $\zeta \in (0, 1)$ if

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,\frac{L}{3}}\| \leq e^{-L^\zeta}. \quad (5.3)$$

(iii) (ω, m, E) -regular for a given $m > 0$ if

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,\frac{L}{3}}\| \leq e^{-m\frac{L}{2}}. \quad (5.4)$$

Remark 5.2. Note that a box $\Lambda_L(x)$ is (ω, θ, E) -suitable if and only if it is (ω, m, E) -regular, where $m = 2\theta \frac{\log L}{L}$. Similarly, $\Lambda_L(x)$ is (ω, ζ, E) -sub-exponentially-suitable if and only if it is $(\omega, 2L^{\zeta-1}, E)$ -regular.

The multiscale analysis converts decay with high probability at a large enough scale into decay with better probabilities at higher scales. We state the strongest version, the bootstrap multiscale analysis of Germinet and Klein [GK1, Theorem 3.4].

Definition 5.3. Let H_ω be a standard ergodic random operator with property (IAD). The multiscale analysis region Σ_{MSA} for H_ω is the set of $E \in \Sigma$ for which there exists some open interval $I \ni E$, such that given any ζ , $0 < \zeta < 1$, and α , $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0 \in 6\mathbb{N}$ and a mass $m > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \dots$, we have

$$\mathbb{P}\{R(m, L_k, I, x, y)\} \geq 1 - e^{-L_k^\zeta} \quad (5.5)$$

for all $k = 0, 1, \dots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$, where

$$\begin{aligned} R(m, L, I, x, y) = & \\ & \{\omega; \text{for every } E' \in I \text{ either } \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is } (\omega, m, E')\text{-regular}\}. \end{aligned} \quad (5.6)$$

Theorem 5.4 ([GK1, Theorem 3.4]). Let H_ω be a standard ergodic random operator with (IAD) and properties (SLI), (NE) and (W) in an open interval \mathcal{I} . Given $\theta > bd$, for each $E \in \mathcal{I}$ there exists a finite scale $\mathcal{L}_\theta(E) = \mathcal{L}_\theta(E, b, d, \varrho)$, bounded on compact subintervals of \mathcal{I} , such that, if for a given $E_0 \in \Sigma \cap \mathcal{I}$ we can verify that

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\omega, \theta, E_0)\text{-suitable}\} > 1 - \frac{1}{841^d} \quad (5.7)$$

at some scale $L_0 \in 6\mathbb{N}$ with $L_0 > \mathcal{L}_\theta(E_0)$, then $E_0 \in \Sigma_{\text{MSA}}$.

Remark 5.5. Explicit estimates on $\mathcal{L}_\theta(E)$ are given in [GK4].

We call Theorem 5.4 the bootstrap multiscale analysis because its proof uses four different multiscale analyses, each one bootstrapping into the next. We present them in the order in which they are used.

Theorem 5.6 ([FK3, Lemma 36], [GK1, Theorem 5.1]). Let H_ω be a standard ergodic random operator with (IAD) and properties (SLI) and (W) in an open interval \mathcal{I} . Let I_0 be a compact subinterval of \mathcal{I} , $E_0 \in I_0$, and $\theta > bd$. Given an odd integer $Y \geq 11$, for any p with $0 < p < \theta - bd$ we can find $\mathcal{Z} = \mathcal{Z}(d, \varrho, Q_{I_0}, \gamma_{I_0}, b, \theta, p, Y)$, such that if for some $L_0 > \mathcal{Z}$, $L_0 \in 6\mathbb{N}$, we have

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\theta, E_0)\text{-suitable}\} > 1 - (3Y - 4)^{-2d}, \quad (5.8)$$

then, setting $L_{k+1} = YL_k$, $k = 0, 1, 2, \dots$, we have that

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\theta, E_0)\text{-suitable}\} \geq 1 - \frac{1}{L_k^p} \quad (5.9)$$

for all $k \geq \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(p, Y, L_0) < \infty$.

The value of Theorem 5.6 is that it requires a very weak starting hypothesis, in which the bound on the probability of the bad event is independent of the scale, and its conclusion, in view of Remark 5.2, gives the starting hypothesis of a modified form of the Dreifus–Klein multiscale analysis, Theorem 5.7 below. Theorem 5.6 is an enhancement of [FK3, Lemma 36], adapted to our assumptions and definitions. It is proven by a multiscale analysis which combines an idea of Spencer [Sp, Theorem 1] with the methods of [DrK].

Theorem 5.7 ([FK3, Theorem 32],[GK1, Theorem 5.2]). *Let H_ω be a standard ergodic random operator with (IAD) and properties (SLI) and (W) in an open interval \mathcal{I} . Let I_0 be a compact subinterval of \mathcal{I} , $E_0 \in I_0$, $\theta > bd$, and $0 < p < \theta - bd$. Then given $p' > p$ and $1 < \alpha < \min\left\{\frac{2p+2d}{p+2d}, \frac{\theta}{p+bd}\right\}$, there is $\mathcal{B} = \mathcal{B}(d, b, \varrho, Q_{I_0}, \gamma_{I_0}, \theta, p, p', \alpha)$, such that, if at some finite scale $L_0 \geq \mathcal{B}$ we verify that*

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (2\theta \frac{\log L_0}{L_0}, E_0)\text{-regular}\} \geq 1 - \frac{1}{L_0^{p'}} , \quad (5.10)$$

then there exists $\delta_1 = \delta_1(d, b, \theta, p, \alpha, L_0) > 0$, such that if we set $I(\delta_1) = [E_0 - \delta_1, E_0 + \delta_1] \cap I_0$, $m_0 = 2\theta \frac{\log L_0}{L_0}$, and $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \dots$, we have

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\frac{m_0}{2}, E)\text{-regular}\} \geq 1 - \frac{1}{L_k^p} \quad \text{for all } E \in I(\delta_1) , \quad (5.11)$$

for all $k = 0, 1, \dots$

If in addition H_ω has property (NE) in \mathcal{I} and we have $\theta > 2p + (b+1)d$, then, fixing a compact subinterval \tilde{I}_0 of \mathcal{I} with $I_0 \subset \tilde{I}_0^\circ$, there is a scale $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}(d, b, \varrho, Q_{\tilde{I}_0}, C_{\tilde{I}_0}, \gamma_{\tilde{I}_0}, \text{dist}(I_0, \mathcal{I} \setminus \tilde{I}_0), \theta, p, p', \alpha)$, such that, if at some finite scale $L_0 \geq \tilde{\mathcal{B}}$ we verify (5.10), we have

$$\mathbb{P}\{R(\frac{m_0}{2}, L_k, I(\delta_1), x, y)\} \geq 1 - \frac{1}{L_k^{2p}} \quad \text{for all } x, y \in \mathbb{Z}^d, |x - y| > L_k + \varrho , \quad (5.12)$$

for all $k = 0, 1, \dots$

Theorem 5.7 is an enhancement of the Dreifus–Klein multiscale analysis [DrK]. The crucial difference is that Theorem 5.7 allows the mass to go to zero as the initial scale L_0 goes to infinity, which may seem very surprising at the first sight. Indeed, in the original versions of the MSA (e.g., [FrS, FrMSS, Dr, DrK, CoH1]), the mass has to be fixed first in order to know how large L_0 has to be chosen. Figotin and Klein [FK3, Theorem 32] were the first to note that the mass may depend on the scale, as in (5.10) above, i.e., a mass proportional to $\frac{\log L_0}{L_0}$. Thus the starting hypothesis (5.10) only requires the decay of the resolvent on finite boxes to be polynomially small in the scale, not exponentially small. Note also that by using the SLI as in (4.33), so we only move between cells, we only need to require $p > 0$ as in [KSS1], not $p > d$ as in [DrK] (we need to consider only the $(3\frac{L}{\ell})^d$ cells that are cores of boxes of side ℓ inside the bigger box of side L , instead of L^d boxes as in [DrK]).

Only the weaker conclusion (5.11) is needed for the bootstrap multiscale analysis; we also stated (5.12) because it is the usual conclusion of this multiscale analysis. Note that for conclusion (5.11) we may take $p' = p$ with $\delta_1 = 0$.

Theorems 5.6 and 5.7 only yield polynomially decaying probabilities for bad events. Germinet and Klein [GK1] introduced new versions of these multiscale analyses that give sub-exponential decay for the probabilities of bad events.

Theorem 5.8. *Let H_ω be a standard ergodic random operator with (IAD) and properties (SLI) and (W) in an open interval \mathcal{I} . Let I_0 be a compact subinterval of \mathcal{I} , $E_0 \in I_0$, and $\zeta_0 \in (0, 1)$. Given an odd integer $Y \geq 11^{\frac{1}{1-\zeta_0}}$, for any ζ_1 with $0 < \zeta_1 < \zeta_0$ we can find $\mathcal{Z} = \mathcal{Z}(d, \varrho, Q_{I_0}, \gamma_{I_0}, b, \zeta_0, \zeta_1, Y)$, such that if for some $L_0 > \mathcal{Z}$, $L_0 \in 6\mathbb{N}$, we have*

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\zeta_0, E_0)\text{-sub-exponentially-suitable}\} > 1 - (3Y - 4)^{-2d}, \quad (5.13)$$

then, setting $L_{k+1} = YL_k$, $k = 0, 1, 2, \dots$, we have that

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\zeta_0, E_0)\text{-sub-exponentially-suitable}\} \geq 1 - e^{-L_k^{\zeta_1}} \quad (5.14)$$

for all $k \geq \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(\zeta_0, \zeta_1, Y, L_0) < \infty$.

Theorem 5.9. *Let H_ω be a standard ergodic random operator with (IAD) and properties (SLI), (NE) and (W) in an open interval \mathcal{I} . Let I_0 be a compact subinterval of \mathcal{I} , $E_0 \in I_0$, \tilde{I}_0 a compact subinterval of \mathcal{I} with $I_0 \subset \tilde{I}_0^\circ$, and $0 < \zeta_2 < \zeta_1 < \zeta_0 < 1$. Then, given $1 < \alpha < \zeta_0/\zeta_1$, there is $\mathcal{C} = \mathcal{C}(d, b, \varrho, Q_{\tilde{I}_0}, C_{\tilde{I}_0}, \gamma_{I_0}, \text{dist}(I_0, \mathcal{I} \setminus \tilde{I}_0), \zeta_0, \zeta_1, \zeta_2, \alpha)$, such that, if at some finite scale $L_0 \geq \mathcal{C}$, $L_0 \in 6\mathbb{N}$, we verify that*

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (2L_0^{\zeta_0-1}, E_0)\text{-regular}\} \geq 1 - e^{-L_0^{\zeta_1}}, \quad (5.15)$$

then there exists $\delta_2 = \delta_2(\zeta_0, \zeta_1, L_0) > 0$ such that, if we set $I(\delta_2) = [E_0 - \delta_2, E_0 + \delta_2] \cap I_0$, $m_0 = 2L_0^{\zeta_0-1}$, and $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \dots$, we have

$$\mathbb{P}\{R\left(\frac{m_0}{2}, L_k, I(\delta_2), x, y\right) \geq 1 - e^{-L_k^{\zeta_2}} \quad (5.16)$$

for all $k = 0, 1, 2, \dots$ and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$.

The equivalent to (5.11) holds in the context of Theorem 5.9, but it will not be needed. In order to get sub-exponential decay of probabilities, the proof of Theorem 5.9 allows the number of bad boxes to grow with the scale.

Outline of the proof of Theorem 5.4. Theorem 5.4 is proven by a bootstrapping argument, making successive use of Theorems 5.6, 5.7, 5.8, and 5.9. We give here an outline of the proof, and refer to [GK1] for the full proof.

- (1) Under the hypotheses of Theorem 5.4, we note that hypothesis (5.8) of Theorem 5.6 is the same as hypothesis (5.7) for appropriate choices of the parameters.
- (2) We apply Theorem 5.6 obtaining a sequence of length scales satisfying conclusion (5.9), with its polynomial decay estimate of the probability of bad events.
- (3) In view of Remark 5.2, it follows that hypothesis (5.10) of Theorem 5.7 is now satisfied at suitably large scale. (We have bootstrapped from hypothesis (5.7) to hypothesis (5.10)!). Thus we can apply Theorem 5.7 with appropriate parameters, getting $\delta_1 > 0$ and a sequence of length scales satisfying conclusion (5.11) for all $E \in I(\delta_1)$. We set $\delta_0 = \delta_1$.

- (4) We fix ζ and α as in Theorem 5.4, and pick $\zeta_0, \zeta_1, \zeta_2$ such that $0 < \zeta < \zeta_2 < \zeta_1 < \zeta_0 < 1 < \alpha < \zeta_0 \zeta_1^{-1} < \zeta_2^{-1} < \zeta^{-1}$. We note that we have bootstrapped again: hypothesis (5.13) of Theorem 5.8 is satisfied at all energies $E \in I(\delta_0)$ at appropriately large scale (the same for all E). Applying Theorem 5.8, we obtain a sequence of length scales for which conclusion (5.14) holds for all $E \in I(\delta_0)$, with its sub-exponential decay estimate of the probability of bad events.
- (5) Using the last part of Remark 5.2, we can see that we have bootstrapped to Theorem 5.9: for any $0 < \zeta_2 < \zeta_1 < \zeta_0 < 1$, hypothesis (5.15) is satisfied at all energies $E \in I(\delta_1)$ at sufficiently large scale (depending on $\zeta_0, \zeta_1, \zeta_2$ but independent of E). We apply Theorem 5.9, obtaining $\delta_2 > 0$ and an exponentially growing sequence of length scales, depending on $\zeta_0, \zeta_1, \zeta_2$, but independent of E , such that conclusion (5.16) holds for all $E \in I(\delta_1)$.
- (6) We have constructed in Step 5 a sequence of length scales for which (5.16) holds for all $E \in I(\delta_0)$. Since the interval $I(\delta_0)$ (which is independent of ζ) can be covered by $[\frac{\delta_0}{\delta_2}] + 1$ closed intervals of length δ_2 , we note that the desired conclusion (5.5) now follows from (5.16), at the energies that are the centers of the $[\frac{\delta_0}{\delta_2}] + 1$ covering intervals, if we take L_0 appropriately large.

□

We will illustrate how to do a multiscale analysis by proving Theorem 5.7 in Section 7, and refer to [GK1] for the proofs of Theorems 5.6, 5.8, and 5.9.

6. FROM THE MULTISCALE ANALYSIS TO LOCALIZATION

The connection between the multiscale analysis and localization is given by the following theorem.

Theorem 6.1. *Let H_ω be a standard ergodic random operator with (IAD) and properties (SGEE) and (EDI) in an open interval \mathcal{I} . Then*

$$\Sigma_{\text{MSA}} \cap \mathcal{I} \subset \Sigma_{\text{EL}} \cap \Sigma_{\text{SSEHSKD}} \cap \mathcal{I}. \quad (6.1)$$

To prove Theorem 6.1 we divide it into Theorems 6.4 and 6.5. Without loss of generality we assume that if properties (GEE), (SGEE), or (EDI) hold, then they hold for every $\omega \in \Omega$.

Lemma 6.2. *Let H_ω be a standard ergodic random operator with properties (GEE) and (EDI) in an open interval \mathcal{I} . Let us fix $m > 0$. For every ω , given $x \in \mathbb{Z}^d$ such that there exists a generalized eigenfunction ψ for H_ω with generalized eigenvalue $E \in \mathcal{I}$ and $\|\chi_x \psi\| \neq 0$, there exists $\tilde{L}(\omega, E, m, x) < \infty$, such that the box $\Lambda_L(x)$ is not (ω, m, E) -regular if $L \geq \tilde{L}(\omega, E, m, x)$.*

Proof. If $x \in \mathbb{Z}^d$ and ψ is a generalized eigenfunction for H_ω with generalized eigenvalue E , and the box $\Lambda_L(x)$ is (ω, m, E) -regular, it follows from (4.34) that for $E \in \mathcal{I}$ we have

$$\begin{aligned} \|\chi_x \psi\| &\leq \tilde{\gamma}_{\{E\}} e^{-m\frac{L}{2}} \|\Gamma_{x,L} \psi\| \leq \tilde{\gamma}_{\{E\}} e^{-m\frac{L}{2}} \|\langle x \rangle^{2\nu} \Gamma_{x,L}\|_\infty \|\psi\|_{\mathcal{H}_-} \\ &\leq d \tilde{\gamma}_{\{E\}} \|\psi\|_{\mathcal{H}_-} L^{d-1} e^{-m\frac{L}{2}} \langle |x| + \frac{L}{2} - 1 \rangle^{2\nu} \\ &\leq 4^\nu d \tilde{\gamma}_{\{E\}} \|\psi\|_{\mathcal{H}_-} \langle x \rangle^{2\nu} L^{d-1} \langle \frac{L}{2} - 1 \rangle^{2\nu} e^{-m\frac{L}{2}}. \end{aligned} \quad (6.2)$$

Since the last expression in (6.2) goes to 0 as $L \rightarrow \infty$, the lemma follows. □

The connection between the multiscale analysis and the generalized eigenfunction expansion is given by the following lemma [GK1, Lemma 4.1].

Lemma 6.3. *Let H_ω be a standard ergodic random operator with properties (GEE) and (EDI) in an open interval \mathcal{I} . Given an open interval I with compact $\bar{I} \subset \mathcal{I}$, $m > 0$, $L \in 6\mathbb{N}$, and $x, y \in \mathbb{Z}^d$, let $R(m, L, I, x, y)$ be as in (5.6). For \mathbb{P} -almost every $\omega \in R(m, L, I, x, y)$, we have*

$$\|\chi_x P_\omega(\lambda) \chi_y\|_2 \leq C \tilde{\gamma}_{\bar{I}} e^{-m\frac{L}{4}} \langle x \rangle^{2\nu} \langle y \rangle^{2\nu}, \quad (6.3)$$

for μ_ω -almost all $\lambda \in I$, with $C = C(m, d, \nu) < +\infty$.

Proof. It follows from (4.9) that

$$\|\chi_x P_\omega(\lambda) \chi_y\|_2 = \|\chi_y P_\omega(\lambda) \chi_x\|_2,$$

for μ_ω -almost every λ , so the roles played by x and y are symmetric.

Let $\omega \in R(m, L, I, x, y)$; then for any $\lambda \in I$, either $\Lambda_L(x)$ or $\Lambda_L(y)$ is (m, λ) -regular for H_ω , say $\Lambda_L(x)$. If $\phi \in \mathcal{H}$, for μ_ω -almost all λ and all $y \in \mathbb{Z}^d$ the vector $P_\omega(\lambda) \chi_y \phi$ is a generalized eigenfunction of H_ω with generalized eigenvalue λ , so for \mathbb{P} -almost every ω it follows from property (EDI) (see (4.34)), using $\chi_x = \chi_{x, \frac{L}{3}} \chi_x$, that

$$\|\chi_x P_\omega(\lambda) \chi_y \phi\| \leq \tilde{\gamma}_{\bar{I}} \|\Gamma_{x, L} R_{\omega, x, L}(\lambda) \chi_{x, \frac{L}{3}}\| \|\Gamma_{x, L} P_\omega(\lambda) \chi_y \phi\|. \quad (6.4)$$

Since $\Lambda_L(x)$ is (m, λ) -regular, we have, using also Lemma 4.1 and the definition of the Hilbert-Schmidt norm, that

$$\|\chi_x P_\omega(\lambda) \chi_y\|_2 \leq \tilde{\gamma}_{\bar{I}} e^{-m\frac{L}{2}} \|\Gamma_{x, L} P_\omega(\lambda) \chi_y\|_2 \quad (6.5)$$

$$\leq \tilde{\gamma}_{\bar{I}} d \left(\frac{3}{2}\right)^{2\nu} L^{d-1} e^{-m\frac{L}{2}} \langle |x| + \frac{L}{2} - 1 \rangle^{2\nu} \langle y \rangle^{2\nu} \quad (6.6)$$

$$\leq \tilde{\gamma}_{\bar{I}} d 3^{2\nu} L^{d-1} e^{-m\frac{L}{2}} \langle \frac{L}{2} - 1 \rangle^{2\nu} \langle x \rangle^{2\nu} \langle y \rangle^{2\nu}, \quad (6.7)$$

so (6.3) follows. \square

Theorem 6.4. *Let H_ω be a standard ergodic random operator with (IAD) and properties (GEE) and (EDI) in an open interval \mathcal{I} . Then*

$$\Sigma_{\text{MSA}} \cap \mathcal{I} \subset \Sigma_{\text{EL}} \cap \mathcal{I}. \quad (6.8)$$

Moreover if $E \in \Sigma_{\text{MSA}} \cap \mathcal{I}$, and we pick an open interval $I \ni E$ and $m > 0$ as in Definition 5.3 with compact $\bar{I} \subset \mathcal{I}$, then for \mathbb{P} -almost every ω , given a generalized eigenfunction Ψ for H_ω with generalized eigenvalue $E' \in I$, we have

$$\limsup_{|x| \rightarrow \infty} \frac{\log \|\chi_x \psi\|}{|x|} \leq -m. \quad (6.9)$$

Proof. Given $E \in \Sigma_{\text{MSA}} \cap \mathcal{I}$, we pick an open interval $I \ni E$ as in Definition 5.3 with compact $\bar{I} \subset \mathcal{I}$. We fix ζ and α such that $0 < \zeta < 1$ and $1 < \alpha < \zeta^{-1}$. By Definition 5.3 there is a scale L_0 and a mass $m > 0$, such that, if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \dots$, then for x and $y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$ we have the estimate (5.5) for $k = 0, 1, 2, \dots$

We will prove that $E \in \Sigma_{\text{EL}}$ by showing that for \mathbb{P} -almost every ω each generalized eigenfunction of H_ω with generalized eigenvalue in I is exponentially decaying in the L^2 -sense. This suffices since for \mathbb{P} -almost every ω we have that μ_ω -almost every $E' \in \mathcal{I}$ is a generalized eigenvalue for H_ω , so we can then conclude that H_ω has pure point spectrum in I .

We fix $b > 1$, to be chosen later. Given $x_0 \in \mathbb{Z}^d$, for each $k = 0, 1, \dots$ we define the discrete annulus

$$A_{k+1}(x_0) = \{\Lambda_{2bL_{k+1}}(x_0) \setminus \Lambda_{2L_k}(x_0)\} \cap \mathbb{Z}^d, \quad (6.10)$$

and the event

$$E_k(x_0) = \{\omega; \Lambda_{L_k}(x_0) \text{ and } \Lambda_{L_k}(x) \text{ are both not } (\omega, m, E')\text{-regular for some } E' \in I \text{ and } x \in A_{k+1}(x_0)\}. \quad (6.11)$$

By (5.5),

$$\mathbb{P}\{E_k(x_0)\} \leq (2bL_{k+1})^d e^{-2L_k^\zeta}, \quad (6.12)$$

and hence

$$\sum_{k=0}^{\infty} \mathbb{P}\{E_k(x_0)\} < \infty, \quad (6.13)$$

so it follows from the Borel-Cantelli Lemma and the countability of \mathbb{Z}^d that

$$\mathbb{P}\{E_k(x_0) \text{ occurs infinitely often for some } x_0 \in \mathbb{Z}^d\} = 0. \quad (6.14)$$

Thus, for \mathbb{P} -almost every ω , given $x_0 \in \mathbb{Z}^d$ there is $k_1(\omega, x_0) \in \mathbb{N}$ such that $\omega \notin E_k(x_0)$ for $k \geq k_1(\omega, x_0)$.

For \mathbb{P} -almost every ω , given a generalized eigenfunction Ψ for H_ω with generalized eigenvalue $E' \in I$, we pick $x_0 \in \mathbb{Z}^d$ such that $\|\chi_{x_0}\psi\| \neq 0$. We set $k_2(\omega, E', x_0) = \min\{k \in \mathbb{N}; L_k \geq \tilde{L}(\omega, E', m, x_0)\}$, where $\tilde{L}(\omega, E', m, x_0)$ is as in Lemma 6.2. Thus, if $k_3(\omega, E', x_0) = \max\{k_1(\omega, x_0), k_2(\omega, E', x_0)\}$, for $k \geq k_3(\omega, E', x_0)$ we conclude that $\Lambda_{L_k}(x)$ is (ω, m, E') -regular for all $x \in A_{k+1}(x_0)$. We pick ρ , with $\frac{1}{3} < \rho < 1$, and $b > \frac{1+\rho}{1-\rho}$, and set

$$\tilde{A}_{k+1}(x_0) = \left\{ \Lambda_{\frac{2b}{1+\rho}L_{k+1}}(x_0) \setminus \Lambda_{\frac{2}{1-\rho}L_k}(x_0) \right\} \cap \mathbb{Z}^d, \quad (6.15)$$

Note that $\tilde{A}_{k+1}(x_0) \subset A_{k+1}(x_0)$ and

$$\text{dist}(x, \mathbb{Z}^d \setminus A_{k+1}(x_0)) \geq \rho|x - x_0| \text{ for all } x \in \tilde{A}_{k+1}(x_0). \quad (6.16)$$

Thus, if $x \in A_{k+1}(x_0)$ with $k \geq k_3(\omega, I, x_0)$, it follows from (4.35) that

$$\|\chi_x\psi\| \leq d\tilde{\gamma}_I L_k^{d-1} e^{-m\frac{L_k}{2}} \|\chi_{x_1}\psi\| \quad (6.17)$$

for some $x_1 \in \Upsilon_L(x)$. If we take $x \in \tilde{A}_{k+1}(x_0)$, we have $x_1 \in A_{k+1}(x_0)$ in view of (6.16), and hence we can apply again (4.35) as in (6.17) to estimate $\|\chi_{x_1}\psi\|$ in terms of some $\|\chi_{x_2}\psi\|$ for some $x_2 \in \Upsilon_L(x_1)$. In fact, it follows from (6.16) that for $x \in \tilde{A}_{k+1}(x_0)$ this procedure can be repeated n times, yielding

$$\|\chi_x\psi\| \leq \left(d\tilde{\gamma}_I L_k^{d-1} e^{-m\frac{L_k}{2}} \right)^n \|\chi_{x_n}\psi\| \quad (6.18)$$

$$\leq \left(\frac{3}{2} \right)^\nu \|\psi\|_{\mathcal{H}_-} \left(d\tilde{\gamma}_I L_k^{d-1} e^{-m\frac{L_k}{2}} \right)^n \langle x_n \rangle^{2\nu} \quad (6.19)$$

for some $x_n \in \mathbb{Z}^d$ with $|x_n - x| \leq n(\frac{L_k}{2} - 1)$, as long as $n(\frac{L_k}{2} - 1) < \rho|x - x_0|$. (We used (4.3) to obtain (6.19)). We thus have the estimate (6.19) with

$$n = \frac{\rho|x - x_0|}{\frac{L_k}{2} - 1} - 1 \geq \frac{\frac{3\rho-1}{2}|x - x_0|}{\frac{L_k}{2} - 1}. \quad (6.20)$$

Note that for all k sufficiently large we have $\frac{L_k}{2} - 1 \geq \frac{L_k}{4}$ and $d\tilde{\gamma}_I L_k^{d-1} e^{-m\frac{L_k}{2}} \leq e^{-m\frac{L_k}{4}}$, in which case it follows from (6.18) and (6.20) that for each $x \in \tilde{A}_{k+1}(x_0)$ we have

$$\|\chi_x \psi\| \leq \left(\frac{3}{2}\right)^\nu \|\psi\|_{\mathcal{H}_-} \langle |x_0| + \rho|x - x_0| \rangle^{2\nu} e^{-\frac{3\rho-1}{2}m|x - x_0|} \quad (6.21)$$

$$\leq 3^\nu \|\psi\|_{\mathcal{H}_-} \langle x_0 \rangle^{2\nu} \langle \rho|x - x_0| \rangle^{2\nu} e^{-\frac{3\rho-1}{2}m|x - x_0|}. \quad (6.22)$$

Thus there exists \tilde{k} , depending only on ρ , d , ν , $\|\psi\|_{\mathcal{H}_-}$, x_0 , L_0 , α , $\tilde{\gamma}_I$, and m , such that if $x \in \tilde{A}_{k+1}(x_0)$ with $k \geq \tilde{k}$ we have (recall $\frac{1}{3} < \rho < 1$)

$$\|\chi_x \psi\| \leq e^{-\frac{\rho(3\rho-1)}{2}m|x - x_0|}. \quad (6.23)$$

Since if $x \in \mathbb{Z}^d$ is such that $|x - x_0| > \frac{L_0}{1-\rho}$, we have $x \in \tilde{A}_{k+1}(x_0)$ for some k , we conclude that there is a finite constant $C_{\psi, \rho}$ such that

$$\|\chi_x \psi\| \leq C_{\psi, \rho} e^{-\frac{\rho(3\rho-1)}{2}m|x - x_0|} \quad \text{for all } x \in \mathbb{Z}^d, \quad (6.24)$$

and hence ψ decays exponentially in the L^2 -sense. In fact, we proved that for each $\frac{1}{3} < \rho < 1$ we have

$$\limsup_{|x| \rightarrow \infty} \frac{\log \|\chi_x \psi\|}{|x|} \leq -\frac{\rho(3\rho-1)}{2}m, \quad (6.25)$$

so letting $\rho \rightarrow 1$ we get (6.9). \square

We now show that the multiscale analysis imply strong sub-exponential HS-kernel decay [GK1, Theorem 3.8]. (Note that for smooth functions of Schrödinger and classical wave operators we always have kernel decay in the deterministic case [GK2, BoGK].)

Theorem 6.5. *Let H_ω be a standard ergodic random operator with (IAD) and properties (SGEE) and (EDI) in an open interval \mathcal{I} . Then*

$$\Sigma_{\text{MSA}} \cap \mathcal{I} \subset \Sigma_{\text{SSEHSDC}} \cap \mathcal{I}. \quad (6.26)$$

Proof. Given $E \in \Sigma_{\text{MSA}} \cap \mathcal{I}$, we pick an open interval $I \ni E$ as in Definition 5.3 with compact $\bar{I} \subset \mathcal{I}$. We will use the generalized eigenfunction expansion (4.11) to show that for any $0 < \xi < 1$ there is a finite constant C_ξ such that

$$\mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_x f(H_\omega) E_\omega(I) \chi_0\|_2^2 \right\} \leq C_\xi e^{-|x|^\xi}, \quad (6.27)$$

for all $x \in \mathbb{Z}^d$, the supremum being taken over all Borel functions f of a real variable, with $\|f\| = \sup_{t \in \mathbb{R}} |f(t)|$. Since our random operator is \mathbb{Z}^d -ergodic, probabilities are translation invariant, so there is no loss of generality in taking $y = 0$.

Given $0 < \xi < 1$, we pick ζ such that $\zeta^2 < \xi < \zeta < 1$ (always possible) and set $\alpha = \frac{\zeta}{\xi}$, note $\alpha < \zeta^{-1}$. By Definition 5.3 there is a scale L_0 and a mass $m_\zeta > 0$, such that, if we set $L_{k+1} = [L_k^\alpha]_{6N}$, $k = 0, 1, \dots$, then for each k we have the estimate (5.5) with $y = 0$ and $x \in \mathbb{Z}^d$ such that $|x| > L_k + \varrho$.

Let us now fix $x \in \mathbb{Z}^d$ and pick k such that $L_{k+1} + \varrho \geq |x| > L_k + \varrho$. In this case Lemma 6.3 asserts that if $\omega \in R(m_\zeta, L_k, I, x, 0)$, then

$$\|\chi_x P_\omega(\lambda) \chi_0\|_2 \leq C_1 e^{-m_\zeta \frac{L_k}{4}} \langle x \rangle^{2\nu} \leq C_1 C_2 e^{-L_k^\zeta}, \quad (6.28)$$

for μ_ω -almost all $\lambda \in I$, with finite constants $C_1 = C_1(m_\zeta, d, \nu, \tilde{\gamma}_I)$ and $C_2 = C_2(\nu, \varrho, \zeta, \xi, m_\zeta)$. We split the expectation in (6.27) in two pieces: where (6.28) holds, and over the complementary event, which has probability less than $e^{-L_k^\zeta}$ by (5.5). From (4.11) we have

$$\begin{aligned} \sup_{\|f\| \leq 1} \|\chi_x f(H_\omega) E_\omega(I) \chi_0\|_2 \\ \leq \sup_{\|f\| \leq 1} \int_I |f(\lambda)| \|\chi_x P_\omega(\lambda) \chi_0\|_2 d\mu_\omega(\lambda) \end{aligned} \quad (6.29)$$

$$\leq \int_I \|\chi_x P_\omega(\lambda) \chi_0\|_2 d\mu_\omega(\lambda). \quad (6.30)$$

Thus, it follows from (6.28) that [with $\mathbb{E}(F(\omega); A) \equiv \mathbb{E}(F(\omega) \chi_A(\omega))$]

$$\begin{aligned} \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_x f(H_\omega) E_\omega(I) \chi_0\|_2^2; R(m_\zeta, L_k, I, x, 0) \right\} \\ \leq C_1^2 C_2^2 \mathbb{E}\{(\mu_\omega(I))^2\} e^{-2L_k^\zeta}. \end{aligned} \quad (6.31)$$

To estimate the second term, note that using (4.7) we have

$$\begin{aligned} \|\chi_x f(H_\omega) E_\omega(I) \chi_0\|_2^2 &\leq \|f\|^2 \|E_\omega(I) \chi_0\|_2^2 \\ &\leq 4^\nu \|f\|^2 \mu_\omega(I), \end{aligned} \quad (6.32)$$

so, using the Schwarz's inequality and (5.5) ,

$$\begin{aligned} \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_x f(H_\omega) E_{H_\omega}(I) \chi_0\|_2^2; \omega \notin R(m_\zeta, L_k, I, x, 0) \right\} \\ \leq 4^\nu [\mathbb{E}\{(\mu_\omega(I))^2\}]^{\frac{1}{2}} e^{-\frac{1}{2}L_k^\zeta}. \end{aligned} \quad (6.33)$$

Since

$$C_3 = C_1^2 C_2^2 \mathbb{E}\{(\mu_\omega(I))^2\} + 4^\nu [\mathbb{E}\{(\mu_\omega(I))^2\}]^{\frac{1}{2}} < \infty \quad (6.34)$$

in view of (4.16), we conclude from (6.31) and (6.33) that (recall $\alpha = \frac{\zeta}{\xi}$)

$$\begin{aligned} \mathbb{E} \left\{ \sup_{\|f\| \leq 1} \|\chi_x f(H_\omega) E_\omega(I) \chi_0\|_2^2 \right\} \\ \leq C_5 e^{-\frac{1}{2}L_k^\zeta} \leq C_3 e^{-\frac{1}{2}L_{k+1}^\zeta} \leq C_3 e^{-\frac{1}{2}(|x| - \varrho)^\xi} \leq C_3 e^{\frac{1}{2}\varrho^\xi} e^{-\frac{1}{2}|x|^\xi} \end{aligned} \quad (6.35)$$

for all $|x| \geq L_0 + \varrho$. Thus (6.27) follows (for a slightly smaller ξ), and Theorem 6.5 is proved. \square

7. HOW TO DO A MULTISCALE ANALYSIS

To exemplify how to perform a multiscale analysis we give the proof of Theorem 5.7, a modification of the proof of [DrK, Theorem 2.2].

Proof of Theorem 5.7. Given $x \in \mathbb{Z}^d$ we set

$$\Xi_{L,\ell}(x) = \Lambda_L(x) \cap \left\{ x + \frac{\ell}{3} \mathbb{Z}^d \right\} \subset \mathbb{Z}^d, \quad \Xi_{L,\ell} = \Xi_{L,\ell}(0), \quad (7.1)$$

$$\mathcal{C}_{L,\ell}(x) = \{ \Lambda_\ell(y); y \in \Xi_{L,\ell}(x), \Lambda_\ell(y) \subset \Lambda_L(x) \}, \quad \mathcal{C}_{L,\ell} = \mathcal{C}_{L,\ell}(0). \quad (7.2)$$

Note $|\Xi_{L,\ell}(x)| \leq (3\frac{L}{\ell} + 1)^d$. By a *cell* we will mean a closed box $\overline{\Lambda}_{\ell/3}(y)$ with $y \in \Xi_{L,\ell}(x)$, the *core* of the box $\Lambda_\ell(y)$. Thus $\mathcal{C}_{L,\ell}(x)$ is the collection of boxes of

side ℓ whose core is a cell and are inside the boundary belt $\tilde{\Upsilon}_L(x)$ of the big box $\Lambda_L(x)$; we have $|\mathcal{C}_{L,\ell}(x)| \leq (3\frac{L}{\ell} - 2)^d$. Note that the big box is covered by cells: $\overline{\Lambda}_L(x) \subset \bigcup_{y \in \Xi_{L,\ell}(x)} \overline{\Lambda}_{\ell/3}(y)$.

Given θ, p, p' such that

$$0 < p < p' < \theta - bd \quad \text{and} \quad 1 < \alpha < \min \left\{ \frac{2p + 2d}{p + 2d}, \frac{\theta}{p + bd} \right\}, \quad (7.3)$$

we pick s and θ' such that

$$\frac{\theta}{2} < \theta' \quad \text{and} \quad p + bd < s < \alpha s < \theta' < \theta. \quad (7.4)$$

Recalling $m_0 = 2\theta \frac{\log L_0}{L_0}$, we have

$$\frac{m_0}{2} < m'_0 = 2\theta' \frac{\log L_0}{L_0} < m_0. \quad (7.5)$$

If $\Lambda_{L_0}(x)$ is (ω, m_0, E_0) -regular and $\text{dist}(\sigma(H_{\omega,x,L_0}), E_0) > L_0^{-s}$, it follows from the (first) resolvent identity that $\Lambda_{L_0}(x)$ is (ω, m'_0, E) -regular for all $E \in I = [E_0 - \delta, E_0 + \delta] \cap I_0$, where

$$\delta = \delta(\theta, \theta', s, L_0) = \frac{1}{2L_0^{2s}} \left(e^{-m'_0 \frac{L_0}{2}} - e^{-m_0 \frac{L_0}{2}} \right). \quad (7.6)$$

Using the hypothesis (5.10) with Remark 5.2, plus property (W) at E_0 with $\eta = L_0^{-s}$ (see (4.37)), we conclude that

$$\begin{aligned} \mathbb{P}\{ \Lambda_{L_0}(0) \text{ is } (\omega, m'_0, E) \text{-regular for every } E \in I \} & \quad (7.7) \\ & \geq 1 - \frac{1}{L_0^{p'}} - \frac{Q_{I_0}}{L_0^{s-bd}} \geq 1 - \frac{1}{L_0^p} \end{aligned}$$

if $L_0 \geq \mathcal{B}_1 = \mathcal{B}_1(d, b, Q_{I_0}, p, p', s)$. Combining with property (IAD), we get that for $L_0 \geq \mathcal{B}_1$ we also have

$$\mathbb{P}\{R(m'_0, L_0, I, x, y)\} \geq 1 - \frac{1}{L_0^{2p}} \quad (7.8)$$

for all $x, y \in \mathbb{Z}^d$ with $|x - y| > L_0 + \varrho$.

We will first prove the weaker conclusion (5.11) by a single energy multiscale analysis which is basically the multiscale analysis of von Dreifus [Dr], except that singular regions are treated as in [DrK]. Let us fix $E \in I$, it obviously follows from (7.7) that

$$\mathbb{P}\{\Lambda_{L_0}(0) \text{ is } (\omega, m'_0, E) \text{-regular}\} \geq 1 - \frac{1}{L_0^p} \quad (7.9)$$

if $L_0 \geq \mathcal{B}_1$. Conclusion (5.11) is proven by induction. Given a scale $L \in 6\mathbb{N}$ and $m > 0$, we let $p_L(m)$ be the probability that a box at scale L is (ω, m, E) -singular (not (ω, m, E) -regular), i.e.,

$$p_L(m) = \mathbb{P}\{\Lambda_L(0) \text{ is } (\omega, m, E) \text{-singular}\}. \quad (7.10)$$

The induction step goes from scale $\ell \geq L_0$ to scale $L = [\ell^\alpha]_{6\mathbb{N}}$: given

$$p_\ell(m) < \frac{1}{\ell^p} \quad \text{with} \quad m = m_\ell \geq 2\theta' \frac{\log \ell}{\ell}, \quad (7.11)$$

we prove

$$p_L(M) < \frac{1}{L^p} \quad \text{for some} \quad M = m_L \geq 2\theta' \frac{\log L}{L}. \quad (7.12)$$

To finish the proof of (5.11), we show $\inf_k m_{L_k} \geq \frac{m_0}{2}$, i.e.,

$$\sum_{k=0}^{\infty} (m_{L_k} - m_{L_{k+1}}) \leq m'_0 - \frac{m_0}{2}. \quad (7.13)$$

The induction step proceeds roughly as in [DrK]. The deterministic part is based on the SLI, but only boxes in $\mathcal{C}_{L,\ell}$ are allowed. The basic idea is that if all boxes in $\mathcal{C}_{L,\ell}$ were (ω, m, E) -regular, then it would follow from applying the estimate (4.33) repeatedly that the big box $\Lambda_L(0)$ is also (ω, M, E) -regular with the difference $m - M$ “small”.

To see how this works, for a given $x \in \mathbb{Z}^d$ we fix $x_0 \in \Xi_{\frac{L+\ell}{3},\ell}(x)$ and apply the SLI estimate (4.33) repeatedly with $\ell' = \ell$, as long as we do not hit the boundary belt $\tilde{\Upsilon}_L(x)$ (see (4.22)). Each time the SLI is performed one gains a factor of $3^d \gamma_I$ and moves to an adjacent cell (see Remark 4.5). After N applications we have

$$\begin{aligned} & \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \\ & \leq (3^d \gamma_I)^N \Pi_{i=1}^N \|\Gamma_{x_i,\ell} R_{\omega,x_i,\ell}(E) \chi_{x_i, \frac{\ell}{3}}\| \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_N, \frac{\ell}{3}}\|, \end{aligned} \quad (7.14)$$

where $x_0, x_1, \dots, x_N \in \Xi_{L,\ell}(x)$ are centers of adjacent cells which are cores of boxes in $\mathcal{C}_{L,\ell}(x)$, i.e., $|x_i - x_{i-1}| = \frac{\ell}{3}$ and $\Lambda_\ell(x_i) \in \mathcal{C}_{L,\ell}(x)$ for $i = 0, 1, \dots, N$. A moment of reflection shows that we are always in this situation as long as

$$(N-1) \frac{\ell}{3} \leq \frac{L-3}{2} - \frac{\ell}{2} - \frac{L+\ell}{6}. \quad (7.15)$$

Since N is an integer, we can always take N to be the unique integer satisfying

$$\frac{L}{\ell} - 3 < N \leq \frac{L}{\ell} - 2. \quad (7.16)$$

If all boxes in $\mathcal{C}_{L,\ell}(x)$ are (ω, m, E) -regular we conclude from (7.14) and (7.16) that

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \leq \left(3^d \gamma_I e^{-m \frac{\ell}{2}}\right)^{\frac{L}{\ell} - 3} \|R_{\omega,x,L}(E)\|. \quad (7.17)$$

Thus,

$$\begin{aligned} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x, \frac{L}{3}}\| & \leq \sum_{x_0 \in \Xi_{\frac{L+\ell}{3},\ell}(x)} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \\ & \leq \left(\frac{L}{\ell} + 2\right)^d \sup_{x_0 \in \Xi_{\frac{L+\ell}{3},\ell}(x)} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \\ & \leq \left(\frac{L}{\ell} + 2\right)^d \left(3^d \gamma_I e^{-m \frac{\ell}{2}}\right)^{\frac{L}{\ell} - 3} \|R_{\omega,x,L}(E)\|. \end{aligned} \quad (7.18)$$

If $\|R_{\omega,x,L}(E)\| \leq L^s$, which holds outside a set of small probability by the Wegner estimate (4.37), we get

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x, \frac{L}{3}}\| \leq L^s \left(\frac{L}{\ell} + 2\right)^d \left(3^d \gamma_I e^{-m \frac{\ell}{2}}\right)^{\frac{L}{\ell} - 3} \equiv e^{-M \frac{L}{2}}, \quad (7.19)$$

with

$$M \geq m \left(1 - \frac{c}{\log \ell}\right) \geq 2\theta' \frac{\log L}{L} \quad (7.20)$$

for ℓ sufficiently large, with c a constant depending only on $d, \gamma_I, \theta', s, \alpha$ and L_0 . The desired estimate (7.13) follows if L_0 is large enough.

Unfortunately the probabilistic estimates do not work. We assumed that all boxes in $\mathcal{C}_{L,\ell}(x)$ are (ω, m, E) -regular and $\|R_{x,L}(E)\| \leq L^s$, thus we can only conclude that

$$\begin{aligned} p_L(M) &\leq (3\frac{L}{\ell} - 2)^d p_\ell(m) + Q_I \frac{1}{L^{s-bd}} \leq (3\frac{L}{\ell} - 2)^d \frac{1}{\ell^p} + Q_I \frac{1}{L^{s-bd}} \\ &\leq \frac{3}{\ell^{p-(\alpha-1)}} + Q_I \frac{1}{L^{s-bd}}. \end{aligned} \quad (7.21)$$

To get $p_L(M) \leq \frac{1}{L^p}$ we would need $p - (\alpha - 1) > p$, which is impossible since $\alpha > 1$.

To fix this problem we must relax the condition that all boxes in $\mathcal{C}_{L,\ell}(x)$ are (ω, m, E) -regular and accept the presence of at least one (ω, m, E) -singular box in $\mathcal{C}_{L,\ell}(x)$. To exploit the independence of events in nonoverlapping boxes (property (IAD)) we will forbid the existence of two *nonoverlapping* singular boxes in $\mathcal{C}_{L,\ell}(x)$.

To see how we obtain the improvement in the probabilities, let us consider the event

$$\begin{aligned} Q_x^{(K)}(E, \ell, L, m) &= \\ \{\omega; \text{ there are } K \text{ nonoverlapping } (\omega, m, E)\text{-singular boxes in } \mathcal{C}_{L,\ell}(x)\}. \end{aligned} \quad (7.22)$$

Using property (IAD) we get

$$\begin{aligned} \mathbb{P}\{Q_x^{(2)}(E, \ell, L, m)\} &\leq |\mathcal{C}_{L,\ell}(x)|^2 p_\ell(m)^2 \leq \left(3\frac{L}{\ell} - 2\right)^{2d} \frac{1}{\ell^{2p}} \leq 9^d \ell^{2d(\alpha-1)} \frac{1}{\ell^{2p}} \\ &< \frac{1}{2\ell^{\alpha p}} \leq \frac{1}{2L^p}, \end{aligned} \quad (7.23)$$

with (7.23) valid for large ℓ if $\alpha < \frac{2p+2d}{p+2d} = 1 + \frac{p}{p+2d}$, which allows for $\alpha > 1$.

We may have fixed one problem but we created another: we cannot estimate the right hand side of (7.14) as before, because we may hit a singular box, i.e., some of the x_i 's in (7.14) may not be the centers of (ω, m, E) -regular boxes. So we must make changes. Taking $\omega \notin Q_0^{(2)}(E, \ell, L, m)$ we exclude the possibility of two nonoverlapping bad boxes in $\mathcal{C}_{L,\ell}(x)$, so if there is one singular box, say $\Lambda_\ell(u)$ (note u depends on ω, ℓ, m, E), to guarantee that $\Lambda_\ell(u') \in \mathcal{C}_{L,\ell}(x)$ is a regular box we need $|u' - u| > \ell + \varrho$. Taking $\ell > 3\varrho$, it suffices to have $|u' - u| > \frac{4\ell}{3}$. Thus $\Lambda_{\frac{4\ell}{3}}(u)$ is our “singular region”, i.e., the region such that boxes in $\mathcal{C}_{L,\ell}(x)$ with cores outside this region are regular. Given $x \in \Xi_{\frac{4\ell}{3}, \ell}$, we estimate $\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x, \frac{\ell}{3}}\|$ by applying the SLI estimate (4.33) repeatedly, as long as we do not hit the boundary belt $\tilde{\Upsilon}_L(0)$, but we now have two cases:

- If $x' \notin \Lambda_{\frac{4\ell}{3}}(u)$ and $\Lambda_\ell(x') \in \mathcal{C}_{L,\ell}(x)$, then x' is the center of a regular box in $\mathcal{C}_{L,\ell}(x)$ and we use (4.33) with $\ell' = \ell$, obtaining

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x', \frac{\ell}{3}}\| \leq 3^d \gamma_I e^{-m\frac{\ell}{2}} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x'', \frac{\ell}{3}}\| \quad (7.24)$$

for some $x'' \in \Upsilon_{\ell,\ell}(x')$, i.e., $|x'' - x'| = \frac{\ell}{3}$.

- If $x' \in \Lambda_{\frac{4\ell}{3}}(u)$ and $\Lambda_{\frac{11\ell}{3}}(u) \subset \Lambda_L(x)$, we apply the SLI estimate (4.33) with $y = x'$, $y' = u$, and $\ell' = 3\ell$, so $k = 9$, obtaining

$$\begin{aligned} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x', \frac{\ell}{3}}\| &\leq 9^d \gamma_I \|\Gamma_{u,3\ell} R_{\omega,u,3\ell}(E) \chi_{x', \frac{\ell}{3}}\| \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x'', \frac{\ell}{3}}\| \end{aligned} \quad (7.25)$$

for some $x'' \in \Upsilon_{3\ell,\ell}(u)$ (see (4.29)), so $|x'' - u| = \frac{4\ell}{3}$, and hence $x'' \notin \Lambda_{\frac{7\ell}{3}}(u)$ with $\Lambda_\ell(x'') \in \mathcal{C}_{L,\ell}(x)$. We are now in the previous case, so we can use (7.24) to get

$$\begin{aligned} & \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x',\frac{\ell}{3}}\| \\ & \leq 27^d \gamma_I^2 e^{-m\frac{\ell}{2}} \|R_{\omega,u,3\ell}(E)\| \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x''',\frac{\ell}{3}}\| \end{aligned} \quad (7.26)$$

for some $x''' \in \Upsilon_{\ell,\ell}(x'')$; note $|x''' - u| \leq \frac{5\ell}{3}$ and $|x''' - x'| \leq \frac{8\ell}{3}$.

To control $\|R_{\omega,u,3\ell}(E)\|$ in (7.26) and $\|R_{\omega,x,L}(E)\|$ in the final expression we will require

$$\|R_{\omega,u,3\ell}(E)\| \leq L^s \text{ for all } u \in \Xi_{L,\ell}(x), \quad (7.27)$$

and

$$\|R_{\omega,0,L}(E)\| \leq L^s. \quad (7.28)$$

To do so, let us define the events

$$\begin{aligned} W_x(E, L, 3\ell, s) = & \\ & \left\{ \omega; \text{ dist}(\sigma(H_{\omega,u,3\ell}), E) > \frac{1}{L^s} \text{ for some } u \in \Xi_{L,\ell}(x) \right\} \end{aligned} \quad (7.29)$$

and

$$W_x(E, L, s) = \left\{ \omega; \text{ dist}(\sigma(H_{\omega,x,L}), E) > \frac{1}{L^s} \right\}, \quad (7.30)$$

We will require $\omega \notin W_x(E, L, 3\ell, s) \cup W_x(E, L, s)$, so (7.27) and (7.28) hold. This will be permissible since it follows from (4.37) that

$$\begin{aligned} \mathbb{P}\{W_x(E, L, 3\ell, s) \cup W_x(E, L, s)\} & \leq (3\frac{L}{\ell} + 1)^d Q_I \frac{(3\ell)^{bd}}{L^s} + Q_I \frac{1}{L^{s-bd}} \\ & < \frac{1}{2\ell^{\alpha p}} \leq \frac{1}{2L^p} \end{aligned} \quad (7.31)$$

for large ℓ , since we chose $s > p + bd$.

Thus if $\omega \notin Q_x^{(2)}(E, \ell, L, m) \cup W_x(E, L, 3\ell, s) \cup W_x(E, L, s)$, for each $x_0 \in \Xi_{\frac{7\ell}{3},\ell}(x)$ we find that after applying either (7.24) or (7.26) with (7.27) repeatedly, stopping before we hit the boundary belt $\tilde{\Upsilon}_L(x)$, we have

$$\begin{aligned} & \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0,\frac{\ell}{3}}\| \\ & \leq \left(3^d \gamma_I e^{-m\frac{\ell}{2}}\right)^{N_r} \left(27^d \gamma_I^2 L^s e^{-m\frac{\ell}{2}}\right)^{N_s} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_N,\frac{\ell}{3}}\| \\ & \leq \left(3^d \gamma_I e^{-m\frac{\ell}{2}}\right)^{N_r} \left(27^d \gamma_I^2 L^s e^{-m\frac{\ell}{2}}\right)^{N_s} L^s, \end{aligned} \quad (7.32)$$

where N_r and N_s are the number of times we used (7.24) or (7.26), respectively, $N = N_r + N_s$. Since $m \geq 2\theta' \frac{\log \ell}{\ell}$ and $\theta' > \alpha s$, we can take ℓ sufficiently large such that

$$27^d \gamma_I^2 L^s e^{-m\frac{\ell}{2}} \leq 27^d \gamma_I^2 \frac{1}{\ell^{\theta' - \alpha s}} < \frac{1}{2}. \quad (7.33)$$

Combining (7.32) and (7.33), we get

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0,\frac{\ell}{3}}\| \leq \left(3^d \gamma_I e^{-m\frac{\ell}{2}}\right)^{N_r} \frac{1}{2^{N_s}} L^s. \quad (7.34)$$

We cannot hit the boundary belt $\tilde{\Upsilon}_L(x)$ as long

$$(N_r - 1)\frac{\ell}{3} \leq \frac{L - 3}{2} - \frac{\ell}{2} - \frac{L + \ell}{6} - \frac{8\ell}{3}, \quad (7.35)$$

where we subtracted $\frac{8\ell}{3}$ due to the fact that we may have gone through the bad region. Thus we always have (7.34) if

$$N_r \leq \frac{L}{\ell} - 10. \quad (7.36)$$

We have then two possible cases: either N_s is large enough so that the right hand side of (7.34) is $\leq e^{-m\frac{L}{2}}L^s$, or we get (7.34) with N_r the integer satisfying

$$\frac{L}{\ell} - 11 < N_r \leq \frac{L}{\ell} - 10, \quad (7.37)$$

and hence

$$\|\Gamma_{x,L}R_{\omega,x,L}(E)\chi_{x_0,\frac{\ell}{3}}\| \leq \left(3^d\gamma_I e^{-m\frac{\ell}{2}}\right)^{\frac{L}{\ell}-11} L^s. \quad (7.38)$$

The estimate (7.38) holds in either case, so we can proceed as in (7.18) to get

$$\|\Gamma_{x,L}R_{\omega,x,L}(E)\chi_{x,\frac{L}{3}}\| \leq L^s \left(\frac{L}{\ell} + 2\right)^d \left(3^d\gamma_I e^{-m\frac{\ell}{2}}\right)^{\frac{L}{\ell}-11} \equiv e^{-M\frac{L}{2}}, \quad (7.39)$$

with

$$M \geq m \left(1 - \frac{c_1}{\log \ell}\right) \geq 2\theta' \frac{\log L}{L} \quad (7.40)$$

for ℓ sufficiently large, with c_1 a constant depending only on $d, \gamma_I, \theta', s, \alpha$ and L_0 . The desired estimate (7.13) follows if L_0 is large enough. Moreover, it follows from (7.23) and (7.31) that for sufficiently large L_0 we have

$$p_L(M) \leq \mathbb{P}\{Q_x^{(2)}(E, \ell, L, m) \cup W_x(E, L, 3\ell, s) \cup W_x(E, L, s)\} < \frac{1}{L^p}. \quad (7.41)$$

The single energy multiscale analysis (5.11) is proven.

We now turn to the proof of the energy interval multiscale analysis (5.12). We fix a compact subinterval \tilde{I}_0 of \mathcal{I} with $I_0 \subset \tilde{I}_0^\circ$, so $\text{dist}(I_0, \mathcal{I} \setminus \tilde{I}_0) > 0$. We require (7.3), (7.4), and

$$\theta > 2p + (b + 1)d. \quad (7.42)$$

As before, the proof proceeds by induction. The initial step in the induction is given by (7.8). Given a scale $L \in 6\mathbb{N}$ and $m > 0$, we set

$$P_L(m, x, y) = \mathbb{P}\{R(m, L, I, x, y)^c\}, \quad (7.43)$$

where A^c denotes the complement of the event A . The induction step goes from scale $\ell \geq L_0$ to scale $L = [\ell^\alpha]_{6\mathbb{N}}$: given that

$$P_\ell(m, x, y) < \frac{1}{\ell^{2p}} \text{ for all } x, y \in \mathbb{Z}^d \text{ with } |x - y| > \ell + \varrho, \text{ with } m = m_\ell \geq 2\theta' \frac{\log \ell}{\ell}, \quad (7.44)$$

we prove

$$P_L(M, x, y) < \frac{1}{L^{2p}} \text{ for all } x, y \in \mathbb{Z}^d \text{ with } |x - y| > \ell + \varrho, \text{ some } M = m_L \geq 2\theta' \frac{\log L}{L}. \quad (7.45)$$

To finish the proof of (5.12), we show that (7.13) holds for these m_{L_k} 's.

The deterministic part of the argument is quite similar to the one we used for the single energy multiscale analysis, except that the probabilistic estimates will require us to accept the possibility of more singular boxes; for every $E \in I$ we will forbid the existence of *four* nonoverlapping singular boxes in either $\mathcal{C}_{L,\ell}(x)$ or $\mathcal{C}_{L,\ell}(y)$. But the probabilistic estimates will require some new ideas.

Let $x \in \mathbb{Z}^d$ and $E \in I$, and suppose there are at most three nonoverlapping (ω, m, E) -singular boxes in $\mathcal{C}_{L,\ell}(x)$, i.e., $\omega \notin Q_x^{(4)}(E, \ell, L, m)$. In this case we can always find three boxes $\Lambda_\ell(u_i) \in \mathcal{C}_{L,\ell}(x)$, $i = 1, 2, 3$, with $|u_i - u_j| > \ell + \varrho$ if $i \neq j$, such that to guarantee that $\Lambda_\ell(u') \in \mathcal{C}_{L,\ell}(x)$ is a (ω, m, E) -regular box we need $|u' - u_i| > \ell + \varrho$ for each $i = 1, 2, 3$. (Note that the u_i depend on ω, ℓ, m, E . We may not need all three boxes, but under our hypothesis it is always true with three.) Taking $\ell > 3\varrho$, it suffices to have $|u' - u_i| > \frac{4\ell}{3}$ for all $i = 1, 2, 3$. We have three cases:

- (1) The closed boxes $\bar{\Lambda}_{\frac{7\ell}{3}}(u_i)$, $i = 1, 2, 3$, are all disjoint. In this case they are the “singular regions”.
- (2) Two of the closed boxes $\bar{\Lambda}_{\frac{7\ell}{3}}(u_i)$, say $i = 1, 2$, are not disjoint, with the third closed box disjoint from the others. In this case we can find $u_{1,2} \in \Xi_{L,\ell}(x)$ such that $\Lambda_{\frac{7\ell}{3}}(u_3)$ and $\Lambda_{5\ell}(u_{1,2})$ are our “singular regions”.
- (3) None of the three closed boxes $\bar{\Lambda}_{\frac{7\ell}{3}}(u_i)$, $i = 1, 2, 3$ is disjoint from the other two. In this case we can find $u_{1,2,3} \in \Xi_{L,\ell}(x)$ such that $\Lambda_{7\ell}(u_{1,2,3})$ is our “singular region”.

The point is that all boxes in $\mathcal{C}_{L,\ell}(x)$ with cores outside the “singular regions” are regular. In all three cases we can find $v_j \in \Xi_{L,\ell}(x)$, $\ell_j \in \{\frac{7\ell}{3}, 5\ell, 7\ell\}$, with $j = 1, \dots, r \leq 3$, $\sum_{j=1}^r \ell_j \leq \frac{22\ell}{3}$, such that the closed boxes $\bar{\Lambda}_{\ell_j}(v_j)$ are disjoint and all boxes in $\mathcal{C}_{L,\ell}(x)$ with cores outside $\bigcup_{j=1}^r \Lambda_{\ell_j}(v_j)$ are (ω, m, E) -regular.

Given $x_0 \in \Xi_{\frac{L+\ell}{3}, \ell}(x)$, we estimate $\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\|$ as before by applying the SLI estimate (4.33) repeatedly, as long as we do not hit the boundary belt $\tilde{\Gamma}_L(x)$. We now have the following cases:

- If $x' \notin \bigcup_{j=1}^r \Lambda_{\ell_j}(v_j)$ and $\Lambda_\ell(x') \in \mathcal{C}_{L,\ell}(x)$, then x' is the center of a regular box in $\mathcal{C}_{L,\ell}(x)$ and we use (7.24).
- If $x' \in \Lambda_{\ell_j}(v_j)$ and $\Lambda_{\ell_j + \frac{4\ell}{3}}(v_j) \sqsubset \Lambda_L(x)$, we apply the SLI estimate (4.33) with $y = x'$, $y' = v_j$, and $\ell' = \ell_j + \frac{2\ell}{3}$, so $k \leq 23$, obtaining

$$\begin{aligned} & \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x', \frac{\ell}{3}}\| \\ & \leq 23^d \gamma_I \|\Gamma_{v_j, \ell_j + \frac{2\ell}{3}} R_{\omega, v_j, \ell_j + \frac{2\ell}{3}}(E) \chi_{x', \frac{\ell}{3}}\| \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x'', \frac{\ell}{3}}\| \end{aligned} \quad (7.46)$$

for some $x'' \in \Upsilon_{\ell_j + \frac{2\ell}{3}, \ell}(v_j)$ (see (4.29)), so $|x'' - v_j| = \frac{\ell_j}{2} + \frac{\ell}{6}$, and hence $x'' \notin \bigcup_{j'=1}^r \Lambda_{\ell_{j'}}(v_{j'})$ with $\Lambda_\ell(x'') \in \mathcal{C}_{L,\ell}(x)$. We are now in the previous case, so we can use (7.24) to get

$$\begin{aligned} & \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x', \frac{\ell}{3}}\| \\ & \leq 69^d \gamma_I^2 e^{-m\frac{\ell}{2}} \|R_{\omega, v_j, \ell_j + \frac{2\ell}{3}}(E)\| \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x''', \frac{\ell}{3}}\| \end{aligned} \quad (7.47)$$

for some $x''' \in \Upsilon_{\ell, \ell}(x'')$; note $|x''' - v_j| \leq \frac{\ell_j + \ell}{2}$ and $|x''' - x'| \leq \ell_j + \frac{\ell}{3}$.

To control $\|R_{\omega, v_j, \ell_j + \frac{2\ell}{3}}(E)\|$ in (7.47) we now require

$$\|R_{\omega, v, \ell'}(E)\| \leq L^s \text{ for all } v \in \Xi_{L,\ell}(x) \text{ and } \ell' \in \{3\ell, \frac{17\ell}{3}, \frac{23\ell}{3}\}, \quad (7.48)$$

i.e.,

$$\omega \notin \bigcup_{\ell' \in \{3\ell, \frac{17\ell}{3}, \frac{23\ell}{3}\}} W_x(E, L, \ell', s). \quad (7.49)$$

Given $x_0 \in \Xi_{\frac{L+\ell}{3}, \ell}(x)$, we apply either (7.24) or (7.47) with (7.48) repeatedly, as long as we do not hit the boundary belt $\tilde{\Upsilon}_L(x)$, obtaining

$$\begin{aligned} & \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \\ & \leq \left(3^d \gamma_I e^{-m\frac{\ell}{2}}\right)^{N_r} \left(69^d \gamma_I^2 L^s e^{-m\frac{\ell}{2}}\right)^{N_s} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_N, \frac{\ell}{3}}\|, \end{aligned} \quad (7.50)$$

where N_r and N_s are the number of times we used (7.24) or (7.47) with (7.48), respectively and $N + N_r + N_s$. Since $m \geq 2\theta' \frac{\log \ell}{\ell}$ and $\theta' > \alpha s$, we can take ℓ sufficiently large such that

$$69^d \gamma_I^2 L^s e^{-m\frac{\ell}{2}} \leq 69^d \gamma_I^2 \frac{1}{\ell^{\theta' - \alpha s}} < \frac{1}{2}. \quad (7.51)$$

Combining (7.50), (7.51), and taking $\omega \notin W_x(E, L, s)$, i.e., $\|R_{\omega,x,L}(E)\| \leq L^s$, we get (7.34), but now to guarantee that we do not hit the boundary belt $\tilde{\Upsilon}_L(x)$ we need

$$(N_r - 1)\frac{\ell}{3} \leq \frac{L - 3}{2} - \frac{\ell}{2} - \frac{L + \ell}{6} - 8\ell, \quad (7.52)$$

where we subtracted 8ℓ due to the fact that we may have gone through the bad regions. Thus we always have (7.34) if

$$N_r \leq \frac{L}{\ell} - 26. \quad (7.53)$$

As before, we have two possibilities: either N_s is large enough so that the right hand side of (7.34) is $\leq e^{-m\frac{\ell}{2}} L^s$, or we get (7.34) with N_r the integer satisfying

$$\frac{L}{\ell} - 27 < N_r \leq \frac{L}{\ell} - 26, \quad (7.54)$$

and hence

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \leq \left(3^d \gamma_I e^{-m\frac{\ell}{2}}\right)^{\frac{L}{\ell} - 27} L^s. \quad (7.55)$$

The estimate (7.55) holds in either case, so we can proceed as in (7.18) to get

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x_0, \frac{\ell}{3}}\| \leq L^s \left(\frac{L}{\ell} + 2\right)^d \left(3^d \gamma_I e^{-m\frac{\ell}{2}}\right)^{\frac{L}{\ell} - 27} \equiv e^{-M\frac{L}{2}} \quad (7.56)$$

with

$$M \geq m \left(1 - \frac{c_2}{\log \ell}\right) \geq 2\theta' \frac{\log L}{L} \quad (7.57)$$

for ℓ sufficiently large, with c_2 a constant depending only on $d, \gamma_I, \theta', s, \alpha$ and L_0 . The desired estimate (7.13) follows if L_0 is large enough.

To finish the proof we need to establish the desired estimate on $P_L(M, x, y)$, where $x, y \in \mathbb{Z}^d$ with $|x - y| > L + \varrho$. Given $u \in \mathbb{Z}^d$, let $Q_u^{(K)}(I, \ell, L, m)$ be the event that there is an energy $E \in I$ for which $\mathcal{C}_{L,\ell}(u)$ contains at least K (ω, m, E) -singular nonoverlapping boxes, i.e.,

$$Q_u^{(K)}(I, \ell, L, m) = \bigcup_{E \in I} Q_u^{(K)}(E, \ell, L, m), \quad (7.58)$$

and let

$$V_u(I, \ell, L, s) = \bigcup_{E \in I} \left[\left(\bigcup_{\ell' \in \{3\ell, \frac{17\ell}{3}, \frac{23\ell}{3}\}} W_u(E, L, \ell', s) \right) \cup W_u(E, L, s) \right]. \quad (7.59)$$

We set

$$Q_{x,y}^{(K)}(I, \ell, L, m) = Q_x^{(K)}(I, \ell, L, m) \cup Q_y^{(K)}(I, \ell, L, m), \quad (7.60)$$

and

$$V_{x,y}(I, \ell, L, s) = V_x(I, \ell, L, s) \cap V_y(I, \ell, L, s). \quad (7.61)$$

If $\omega \notin Q_{x,y}^{(4)}(I, \ell, L, m) \cup V_{x,y}(I, \ell, L, s)$, for every $E \in I$ we have (7.56) and (7.57) for either $\Lambda_L(x)$ or $\Lambda_L(y)$, and hence, using the translation invariance of the probabilities, we have

$$P_L(M, x, y) \leq 2\mathbb{P}\{Q_0^{(4)}(I, \ell, L, m)\} + \mathbb{P}\{V_{x,y}(I, \ell, L, s)\}. \quad (7.62)$$

We first estimate $\mathbb{P}\{Q_0^{(4)}(I, \ell, L, m)\}$. Let $\mathcal{C}_{L,\ell}^{(K)}$ denote the collection of K nonoverlapping boxes in $\mathcal{C}_{L,\ell}$. We have, using property (IAD) and the induction hypothesis, that

$$\begin{aligned} & \mathbb{P}\{Q_0^{(4)}(I, \ell, L, m)\} \\ & \leq \sum_{\{\Lambda_\ell(u), \Lambda_\ell(v)\} \in \mathcal{C}_{L,\ell}^{(2)}} \mathbb{P}\{R(m, \ell, I, u, v)^c\} \mathbb{P} \left\{ \bigcup_{\{\Lambda_\ell(u'), \Lambda_\ell(v')\} \in \mathcal{C}_{L,\ell}^{(2)}} R(m, \ell, I, u', v')^c \right\} \\ & \leq \sum_{\{\Lambda_\ell(u), \Lambda_\ell(v)\} \in \mathcal{C}_{L,\ell}^{(2)}} \mathbb{P}\{R(m, \ell, I, u, v)^c\} \mathbb{P} \left\{ \bigcup_{\{\Lambda_\ell(u'), \Lambda_\ell(v')\} \in \mathcal{C}_{L,\ell}^{(2)}} R(m, \ell, I, u', v')^c \right\} \\ & \leq \left(\sum_{\{\Lambda_\ell(u), \Lambda_\ell(v)\} \in \mathcal{C}_{L,\ell}^{(2)}} \mathbb{P}\{R(m, \ell, I, u, v)^c\} \right)^2 = \left(\sum_{\{\Lambda_\ell(u), \Lambda_\ell(v)\} \in \mathcal{C}_{L,\ell}^{(2)}} P_\ell(m, u, v) \right)^2 \\ & \leq \left(\left(3 \frac{L}{\ell} \right)^{2d} \frac{1}{\ell^{2p}} \right)^2 \leq 3^{4d} \frac{1}{\ell^{4(p-d(\alpha-1))}}. \end{aligned} \quad (7.63)$$

It remains to estimate $\mathbb{P}\{V_{x,y}(I, \ell, L, s)\}$. Let $\tilde{\sigma}(A) = \sigma(A) \cap \tilde{I}_0$ for any operator A . If $\Lambda_{\ell_1}(u)$ and $\Lambda_{\ell_2}(v)$ are nonoverlapping boxes, then it follows from properties (IAD), (NE) and (W) that for $\eta < \text{dist}(I_0, \mathcal{I} \setminus \tilde{I}_0)$ we have

$$\mathbb{P}\{\text{dist}(\tilde{\sigma}(H_{\omega,u,\ell_1}), \tilde{\sigma}(H_{\omega,v,\ell_2})) \leq \eta\} \leq C_{\tilde{I}_0} Q_{\tilde{I}_0} \eta \ell_1^{bd} \ell_2^d. \quad (7.64)$$

To see that, let \mathcal{F}_1 and \mathcal{F}_2 be the σ -algebras generated by events based on the boxes $\Lambda_{\ell_1}(u)$ and $\Lambda_{\ell_2}(v)$, respectively. We set \mathbb{P}_i to be the restriction of the probability measure \mathbb{P} to \mathcal{F}_i , with \mathbb{E}_i the corresponding expectation and ω_i the corresponding variable of integration, $i = 1, 2$. Using the independence given by property (IAD), we have

$$\begin{aligned} & \mathbb{P}\{\text{dist}(\tilde{\sigma}(H_{\omega,u,\ell_1}), \tilde{\sigma}(H_{\omega,v,\ell_2})) \leq \eta\} = \\ & \mathbb{E}_2 \{\mathbb{P}_1 \{\text{dist}(\tilde{\sigma}(H_{\omega_1,u,\ell_1}), \tilde{\sigma}(H_{\omega_2,v,\ell_2})) \leq \eta\}\} \end{aligned} \quad (7.65)$$

For a fixed ω_2 we have $\tilde{\sigma}(H_{\omega_2, v, \ell_2}) = \{\lambda_1, \lambda_2, \dots, \lambda_{N_{\omega_2}}\}$, where

$$\mathbb{E}_2(N_{\omega_2}) \leq C_{\tilde{I}_0} \ell_2^d \quad (7.66)$$

by property (NE). (Note that N_{ω_2} and $\lambda_1, \lambda_2, \dots, \lambda_{N_{\omega_2}}$ depend on ω_2, v, ℓ_2 .) Using property (W), we get

$$\begin{aligned} \mathbb{P}_1\{\text{dist}(\tilde{\sigma}(H_{\omega_1, u, \ell_1}), \tilde{\sigma}(H_{\omega_2, v, \ell_2})) \leq \eta\} &\leq \\ \sum_{j=1}^{N_{\omega_2}} \mathbb{P}_1\{\text{dist}(\tilde{\sigma}(H_{\omega_1, u, \ell_1}), \lambda_j) \leq \eta\} &\leq Q_{\tilde{I}_0} \eta \ell_1^{bd} N_{\omega_2}. \end{aligned} \quad (7.67)$$

The estimate (7.64) follows from (7.65), (7.67), and (7.66).

Let $Z_{x,y}(I, \ell, L, s)$ denote the event that

$$\text{dist}(\tilde{\sigma}(H_{\omega, u, \ell_1}), \tilde{\sigma}(H_{\omega, v, \ell_2})) \leq \frac{2}{L^s} \quad (7.68)$$

for either

- (i) $u = x, v = y$, and $\ell_1 = \ell_2 = L$, or
- (ii) $u = x, \ell_1 = L$, and some $v \in \Xi_{L, \ell}(y)$ and $\ell_2 \in \{3\ell, \frac{17\ell}{3}, \frac{23\ell}{3}\}$, or
- (iii) $v = y, \ell_2 = L$, and some $u \in \Xi_{L, \ell}(x)$, and $\ell_1 \in \{3\ell, \frac{17\ell}{3}, \frac{23\ell}{3}\}$, or
- (iv) some $u \in \Xi_{L, \ell}(x), v \in \Xi_{L, \ell}(y)$, and $\ell_1, \ell_2 \in \{3\ell, \frac{17\ell}{3}, \frac{23\ell}{3}\}$.

Clearly

$$V_{x,y}(I, \ell, L, s) \subset Z_{x,y}(I, \ell, L, s), \quad (7.69)$$

and it follows from (7.64), if L_0 is large enough so $\frac{1}{L_0^s} \leq \frac{1}{L_0^s} < \text{dist}(I_0, \mathcal{I} \setminus \tilde{I}_0)$, that

$$\begin{aligned} &\mathbb{P}\{Z_{x,y}(I, \ell, L, s)\} \\ &\leq \frac{2C_{\tilde{I}_0} Q_{\tilde{I}_0}}{L^s} \left\{ L^{(b+1)d} + 6 \left(3\frac{L}{\ell}\right)^d L^d \left(\frac{23\ell}{3}\right)^{bd} + \left(3\frac{L}{\ell}\right)^{2d} \left(\frac{23\ell}{3}\right)^{(b+1)d} \right\} \\ &\leq \frac{C_{d,b,\alpha} C_{\tilde{I}_0} Q_{\tilde{I}_0}}{L^s} \left\{ L^{(b+1)d} + L^{(2+\frac{b-1}{\alpha})d} \right\} \leq \frac{2C_{d,b,\alpha} C_{\tilde{I}_0} Q_{\tilde{I}_0}}{L^{s-(b+1)d}}, \end{aligned} \quad (7.70)$$

where $C_{d,b,\alpha}$ is a finite constant depending only on d, b, α .

It now follows from (7.62), (7.63), and (7.70) that

$$P_L(M, x, y) \leq 2 \cdot 3^{4d} \frac{1}{\ell^{4(p-d(\alpha-1))}} + \frac{4C_{d,b,\alpha} C_{\tilde{I}_0} Q_{\tilde{I}_0}}{L^{s-(b+1)d}} < \frac{1}{L^{2p}} \quad (7.71)$$

for sufficiently large L , since $\alpha < \frac{2p+2d}{p+2d}$ and $s > 2p + (b+1)d$.

□

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